Minnesota Fats and Eddie Felson
The Hustler (1961)

Ben-Hur (1959)

A more general *inverse* problem:

Can you chisel a stone that would roll along an *arbitrary* periodic trajectory?

Such stones called TRAJECTOIDS almost always exist

- a mathematical property of the rotation group
- with some potential applications

Yaroslav SOBOLEV • Jean-Pierre ECKMANN

Ruoyu DONG • Steve GRANICK • Bartosz GRZYBOWSKI

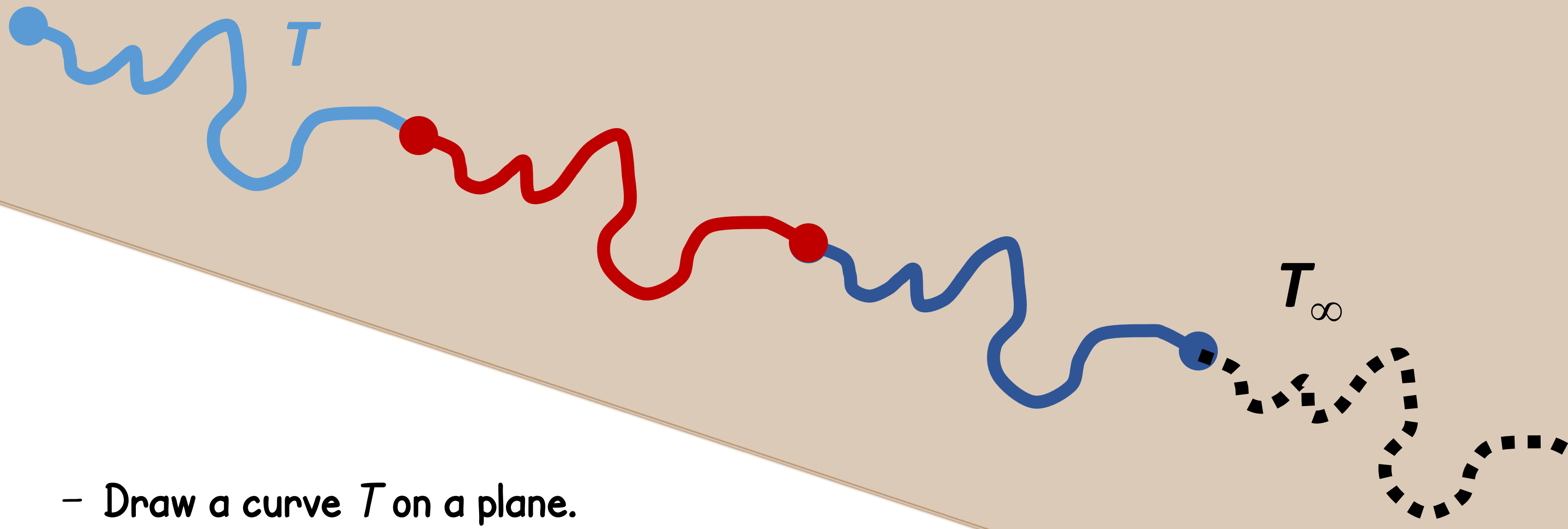
Eckmann et al. Tumbling downhill along a given curve. *Amer Math Soc Notices* - in press.

Sobolev et al. Solid-body trajectoids shaped to roll along desired pathways. *Nature* 2023.

OUTLINE

- Introduction - the rolling problem
- Finite - fast
- Infinite - main result
- This transcends from rolling to the general context rotation group\parallel transport
- There are natural application in anything that rotates…

THE BASIC SETTING

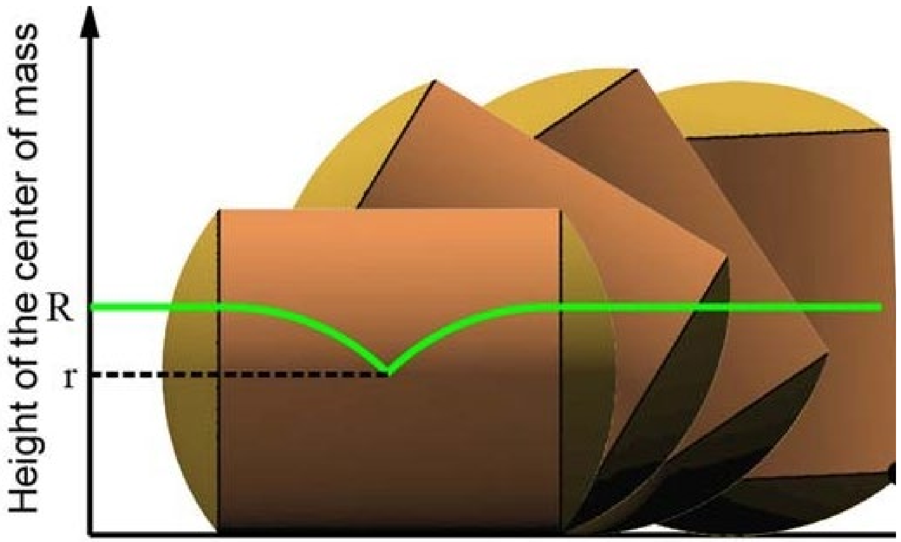


- Draw a curve T on a plane.
- Copy it to form an infinite periodic trajectory T_∞
- Incline the plane slightly.

Find a solid body – a 'trajectoid' – that rolls exactly along T_∞ without slipping or pivoting (spinning).

The simplest trajectoid is the cylinder

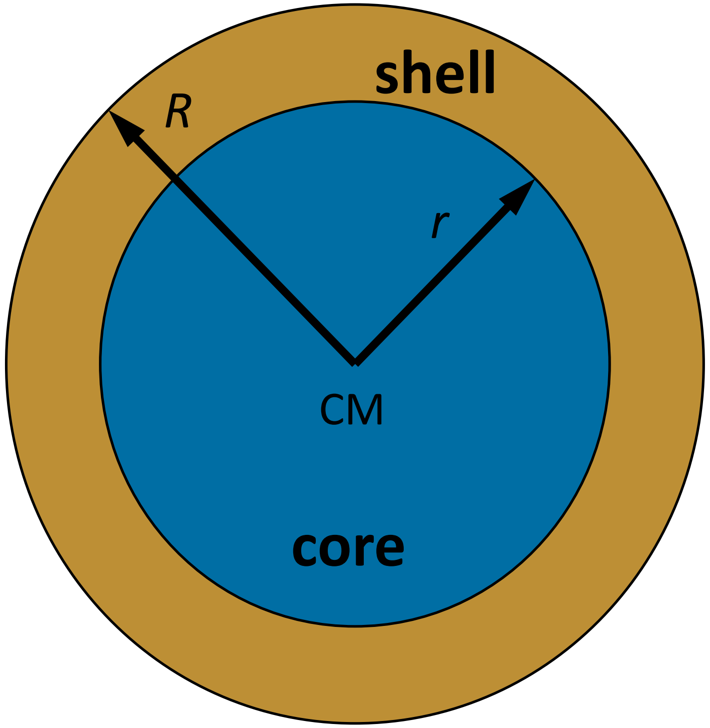
that rolls along a straight line

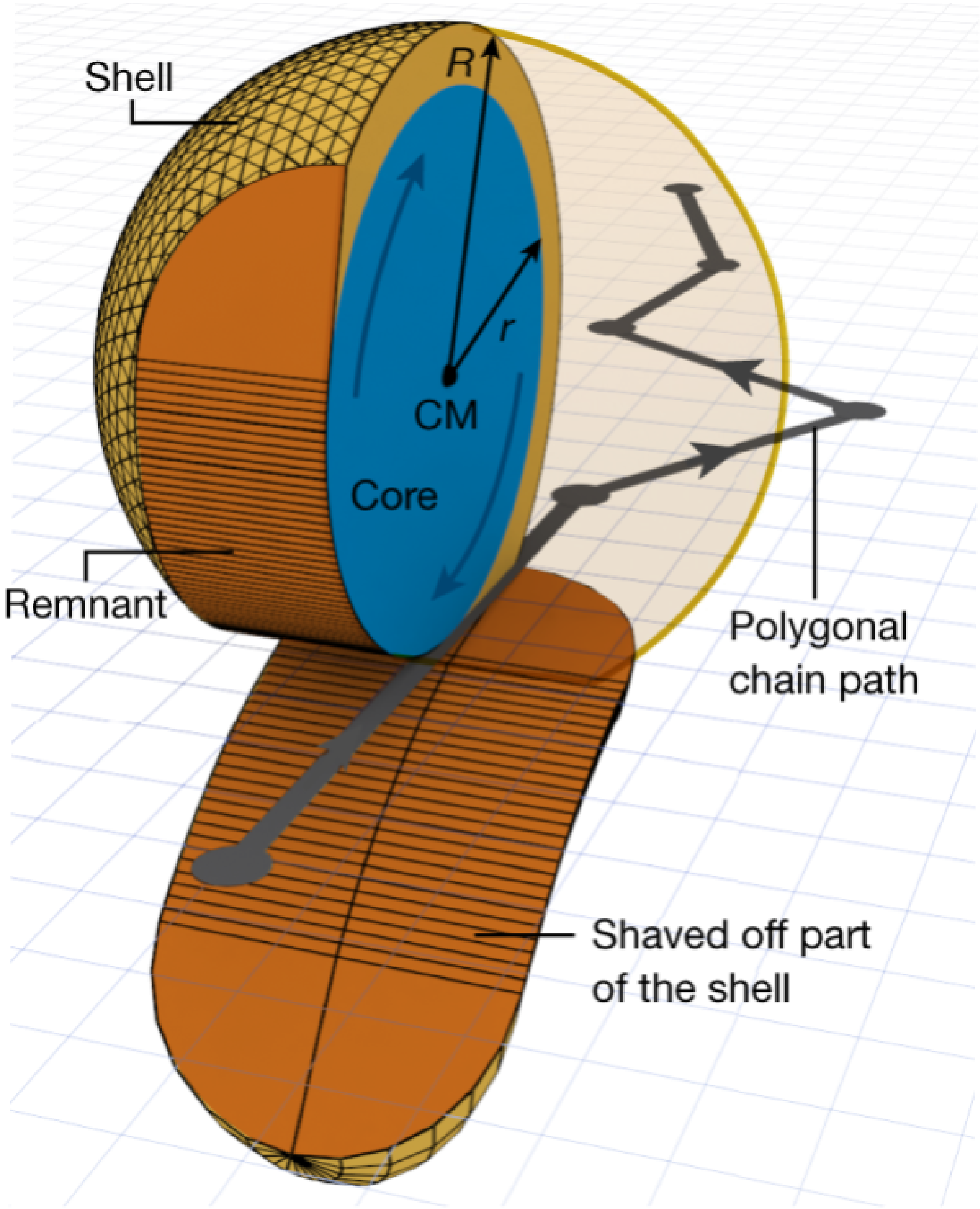


FINITE PATH

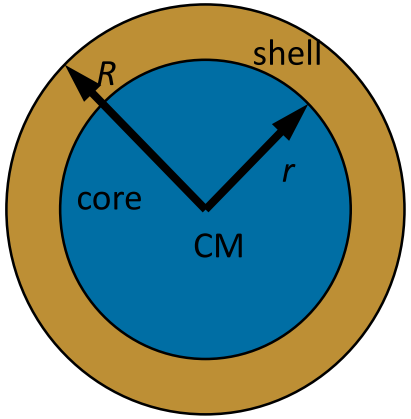
Solid that rolls along a general curvilinear path

- a trajectoid -can be obtained by "shaving"



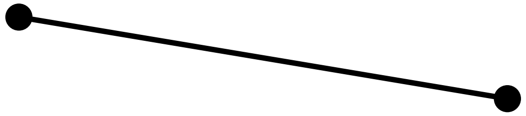


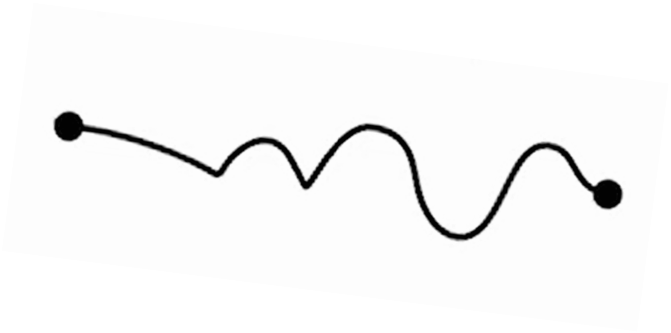
Please ask questions during the talk

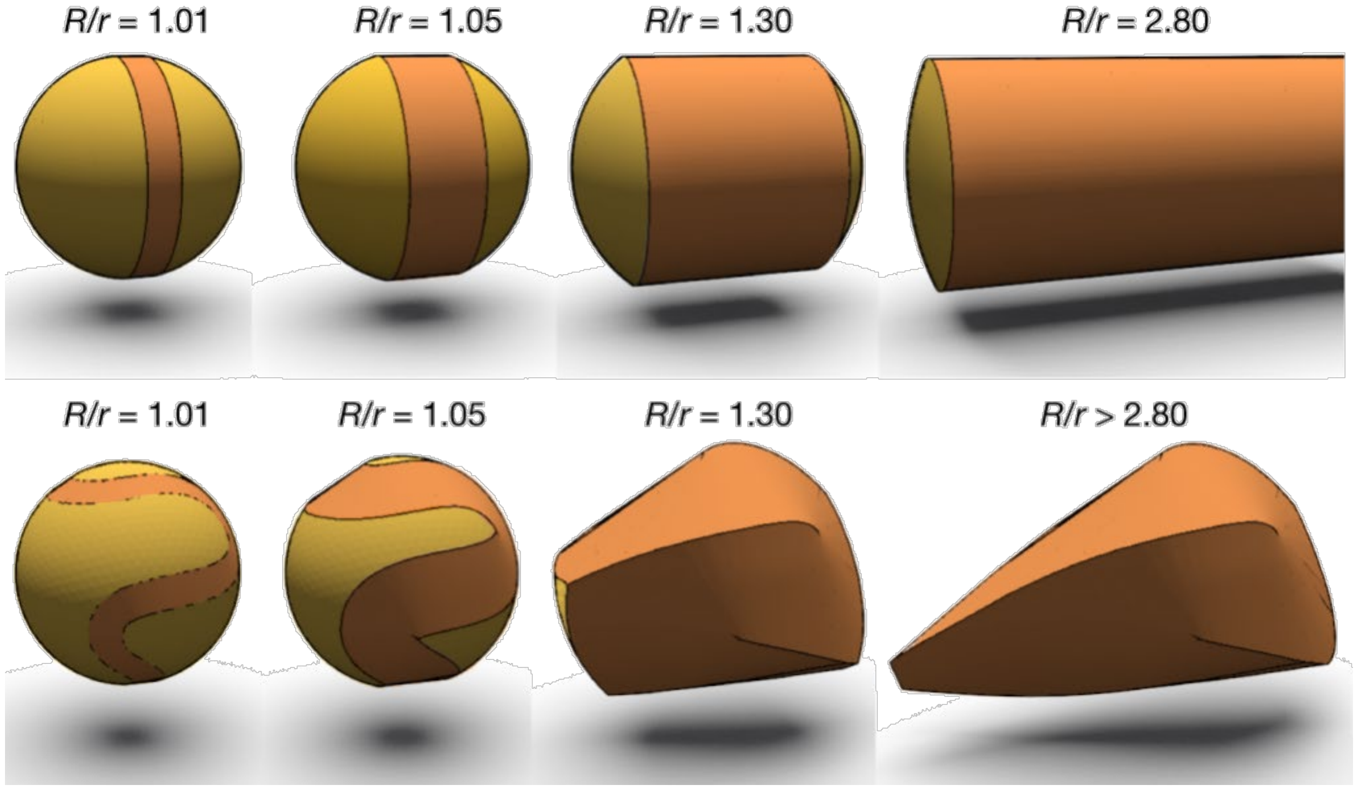


The heavy core of the trajectoid traces the path

while the locally cylindrical shell constrains it to the path

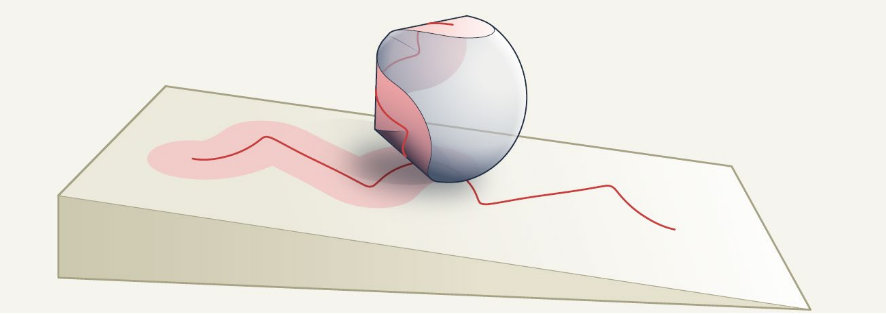




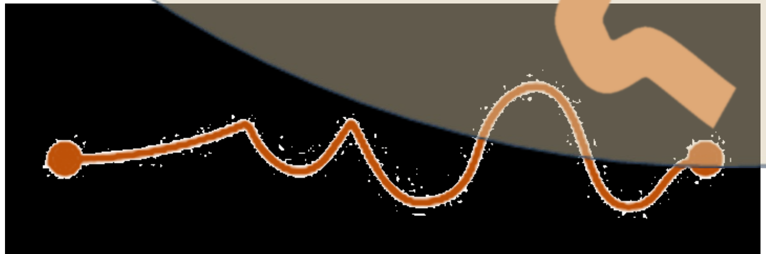


Thin shell → Thick shell

FINITE PATH

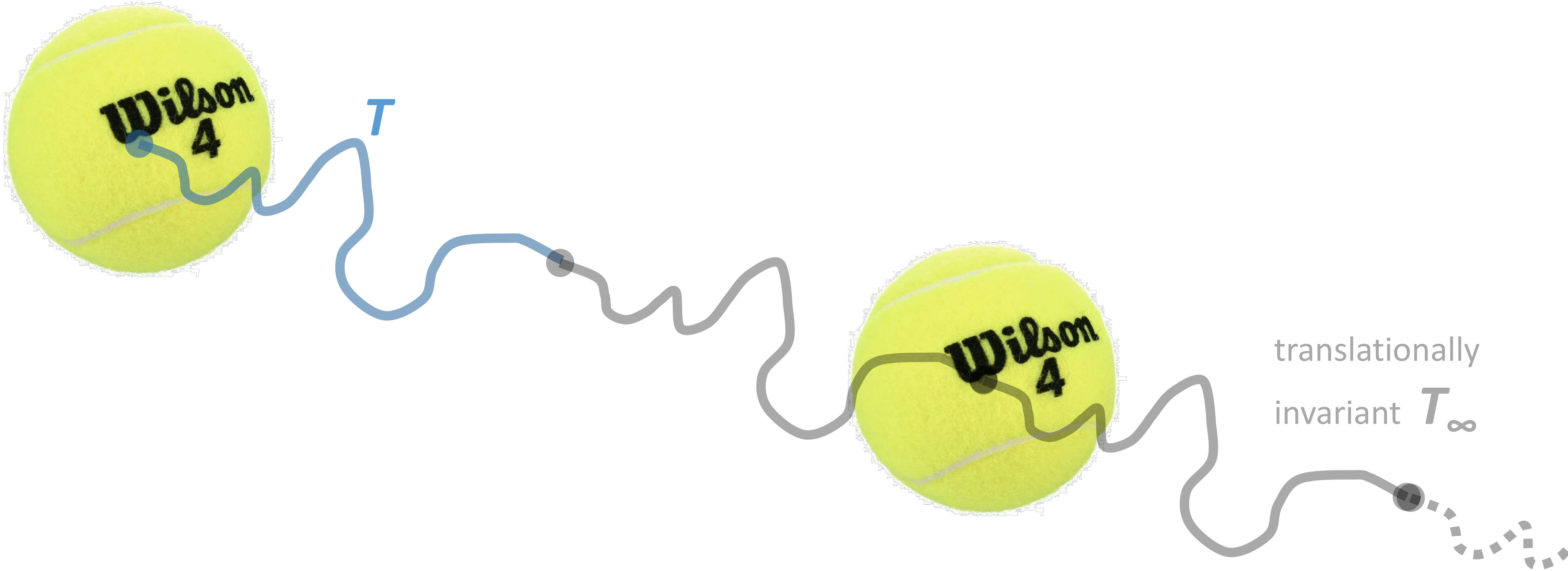


Limit $r \to \infty$



INFINITE PERIODIC PATH

A trajectoid must periodically regain its initial 3D orientation

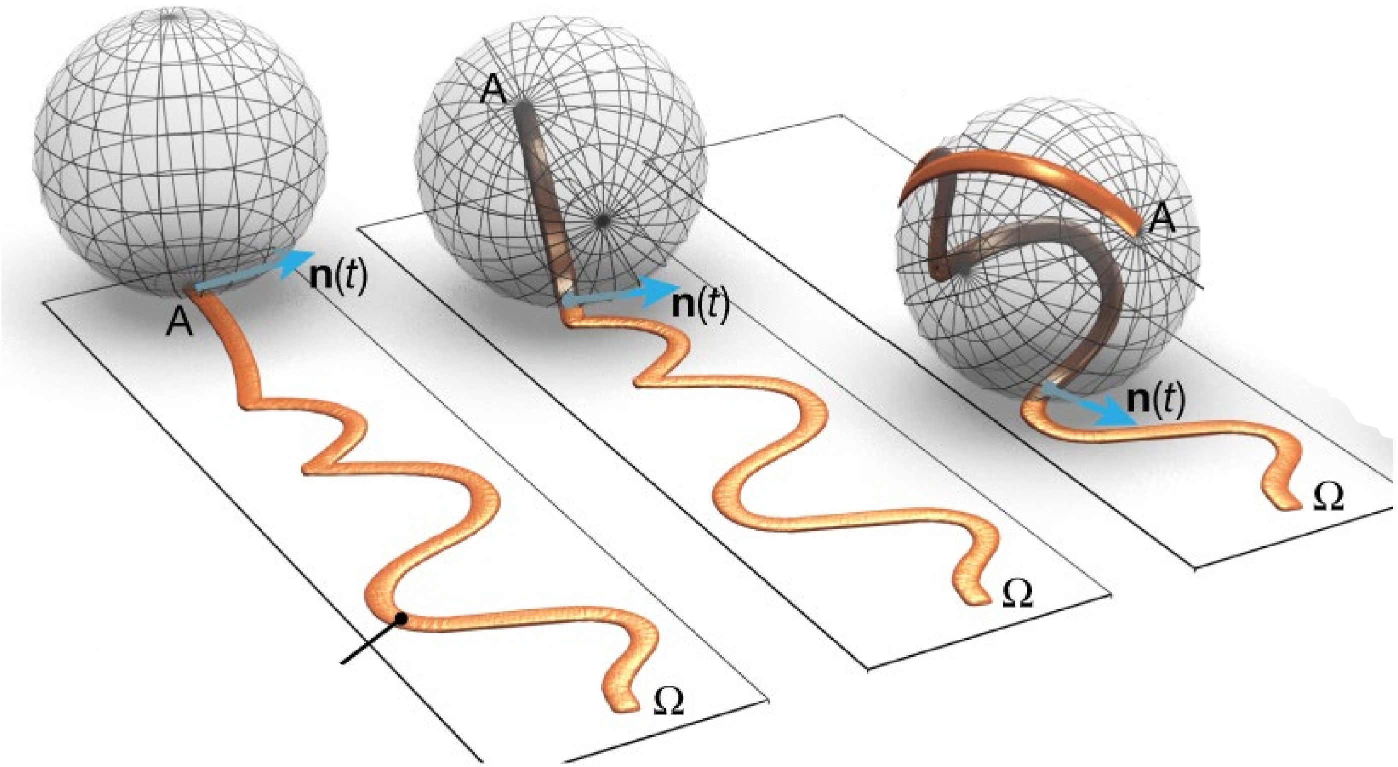


n-period trajectoid exists for T_∞ if there is a ball $r > 0$ (its core) that

after rolling along n periods regains its original orientation (without slipping or pivoting)

It is natural to discuss the trajectoid problem in terms of rotations acting on the heavy core

Planar path defined by normal $\mathbf{n}(t) = \mathbf{A\Omega}$



Each segment δt induces a rotation by δt/r around n(t)

$$R_i = \exp\left(\frac{\delta t}{r}\mathbf{n}(t_i)\cdot\vec{\mathbf{L}}\right)$$

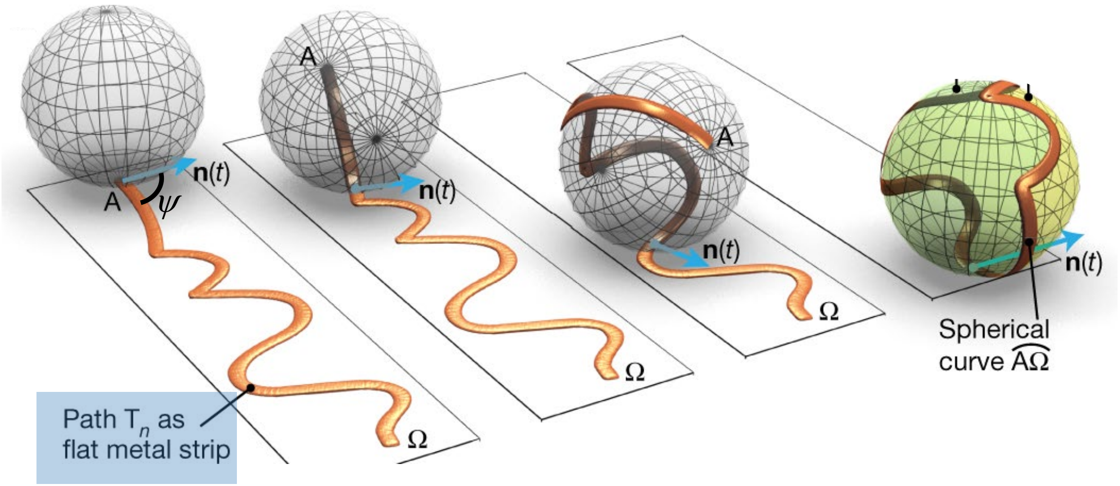
Regaining orientation

$$R_{\mathrm{A}\Omega} = \prod_i R_i = \mathcal{T}\left[\exp\left(\frac{1}{r}\int_{\mathrm{A}}^{\Omega} dt\,\mathbf{n}(t)\cdot\vec{\mathbf{L}}\right)\right] = 1$$

rotation group generators

$$L_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad L_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad L_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Tracing the path on the sphere conserves the geodesic (in-plane) curvature



normal $\quad \mathbf{n}(t) = \left(\cos\psi(t), \sin\psi(t), 0\right)$

curvature $\quad \kappa(t) = \dfrac{d\psi}{dt}$

regaining

orientation $\Leftrightarrow$ $\begin{cases} \text{same contact point} \\ \text{(path } \mathrm{A\Omega} \text{ closes on sphere)} \\ + \\ \text{same orientation w.r.t. } \mathbf{n}(t) \end{cases}$

$R_{\mathrm{A\Omega}} = \prod R_i = 1$

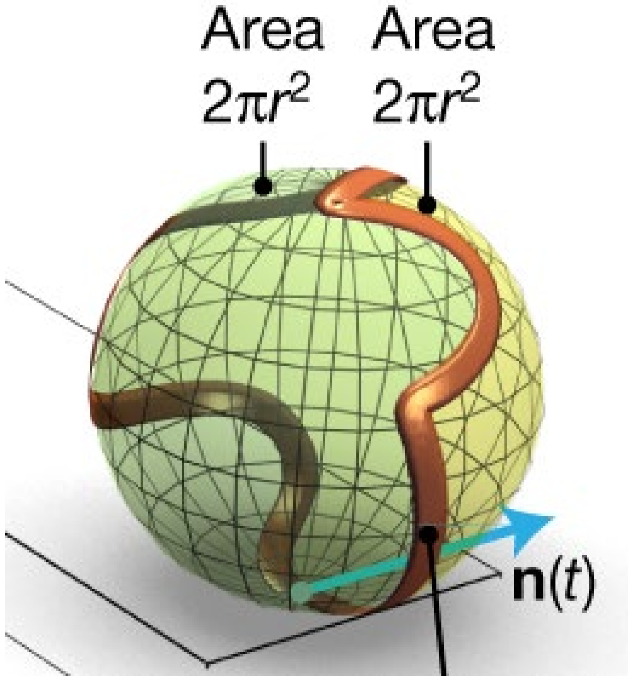
$\Rightarrow$ index of curvature κ vanishes

$$\psi_\Omega - \psi_\mathrm{A} = \int_\mathrm{A}^\Omega dt\, \kappa(t) = 0$$

(for self-intersecting paths add $\pm 2\pi$ for each loop)

Please ask questions during the talk

Path of a trajectoid encloses half of the sphere area



From Gauss-Bonnet theorem

$$\int_{\text{manifold}} ds \left(\begin{array}{c}\text{Gaussian}\\ \text{curvature}\end{array}\right) + \int_{\text{boundary}} dt\,\kappa(t) = 2\pi \left(\begin{array}{c}\text{Euler}\\ \text{characteristic}\end{array}\right)$$

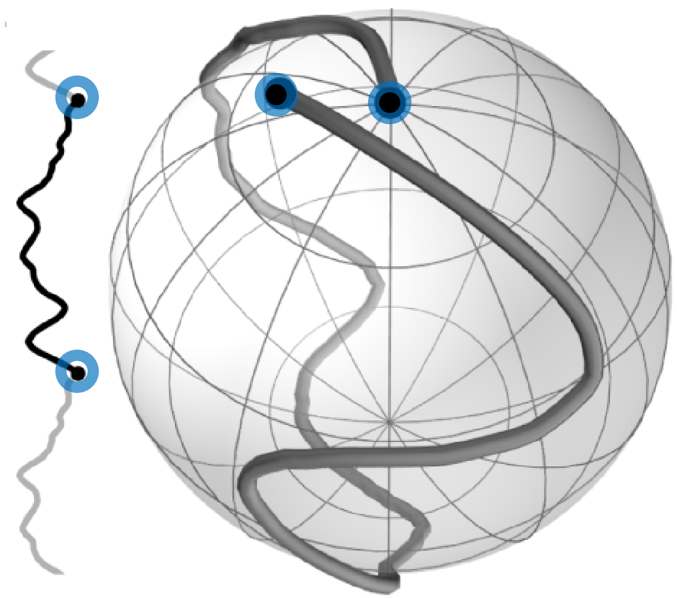
applied to a sphere

$$\frac{S}{r^2} + \cancel{\int_{\text{boundary}} dt\,\kappa(t)} = 2\pi$$

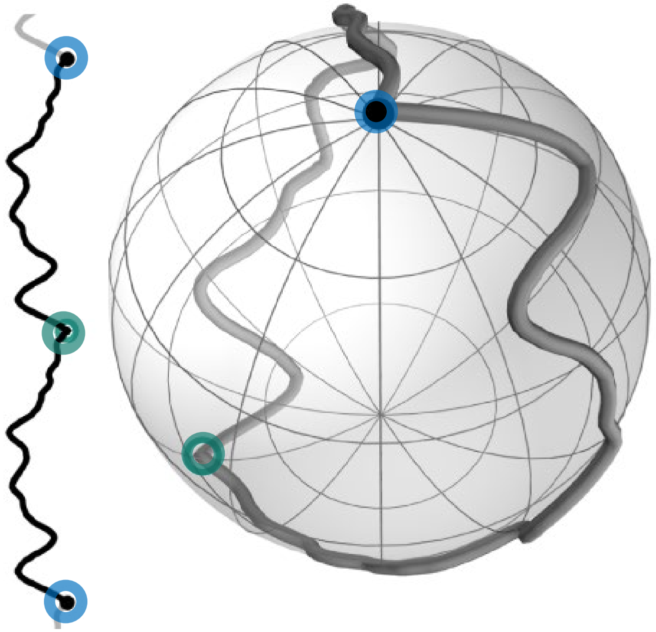
Invariance of geodesic curvature

(think of metal strip)

The difficulty of finding a trajectoid sharply depends on the number of periods:

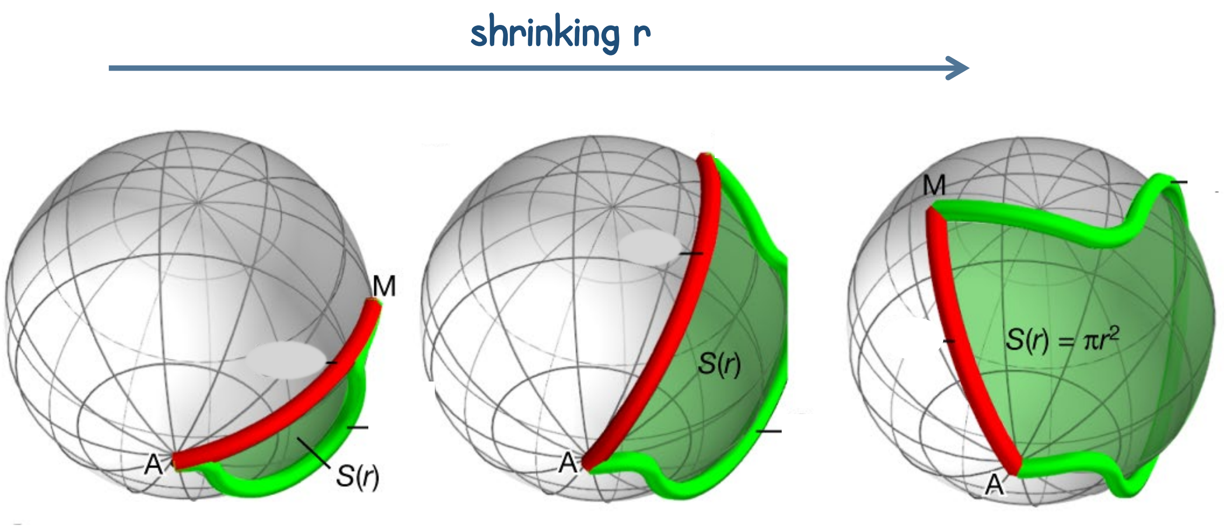


1-period trajectories
almost never exist



2-period trajectories
almost always exist

The TPT theorem (two-period trajectoid) explains their generic existence



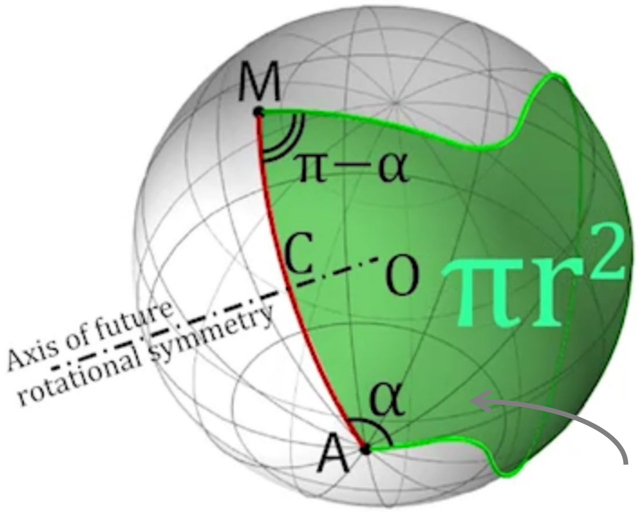
<u>Definition:</u> Path T_∞ has the property π

if rolling a sphere $r > 0$ along the period T, together

with a great arc, encloses an area $S = \pm\pi\, r^2$.

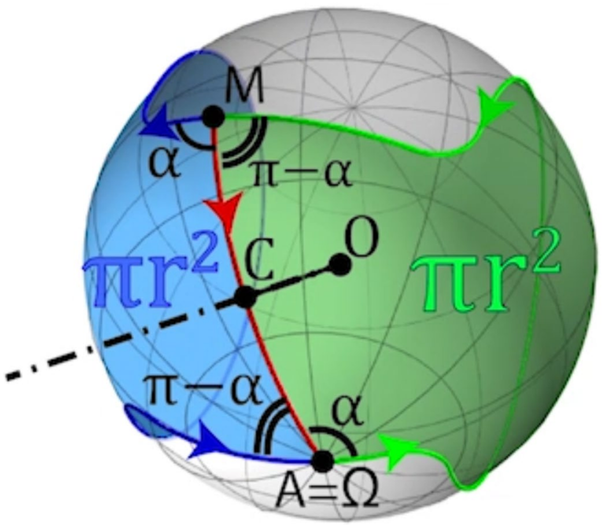
The TPT theorem:

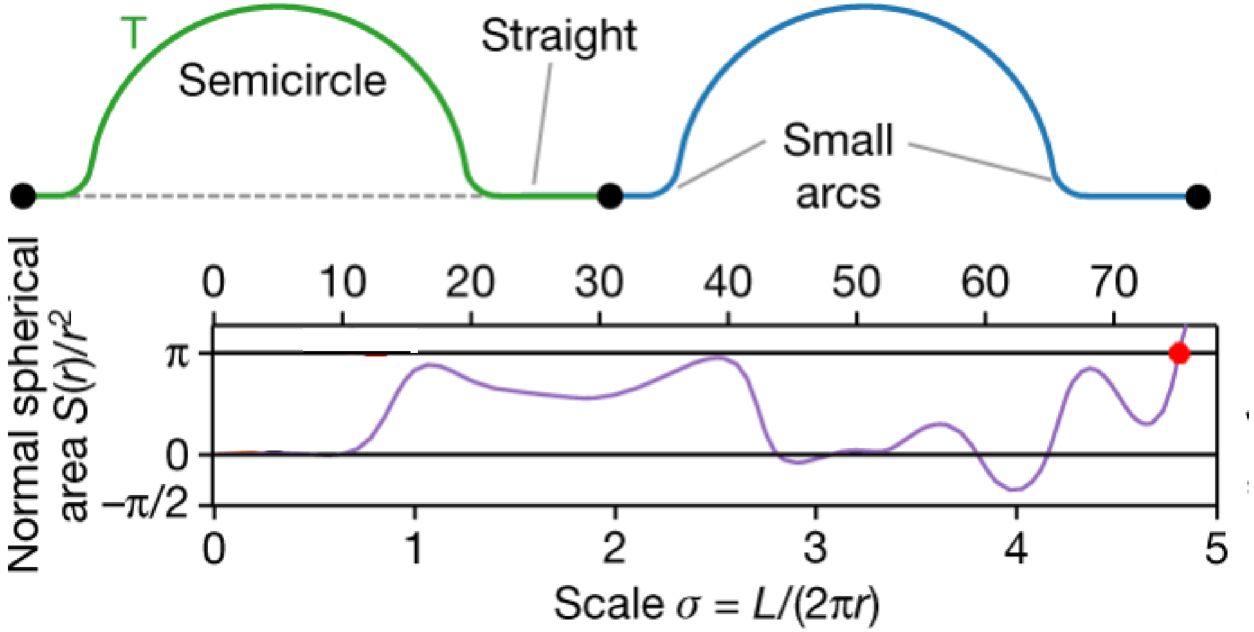
A two-period trajectoid (TPT) can be constructed for T that has the property π.



GB theorem:

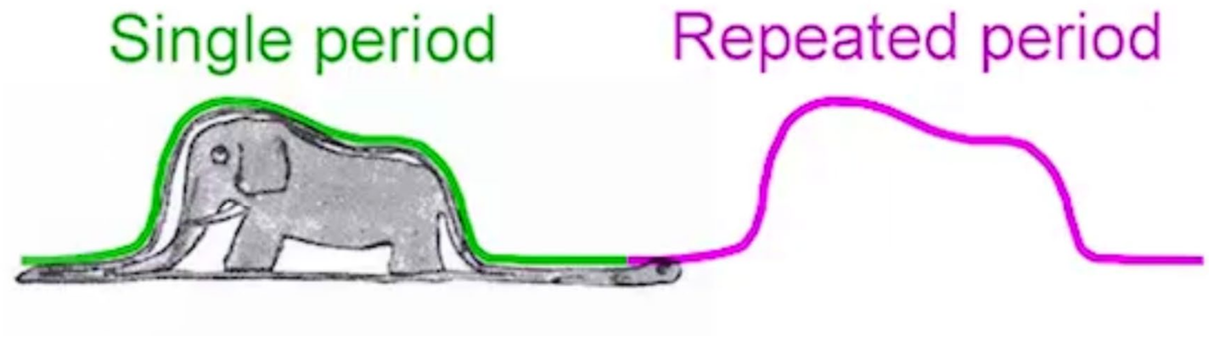
$$\int_{\text{boundary}} dt\,\kappa(t) = \pi$$

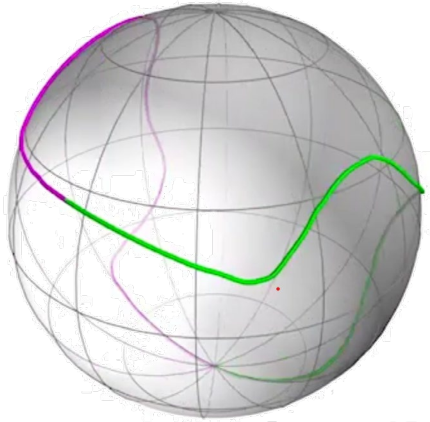


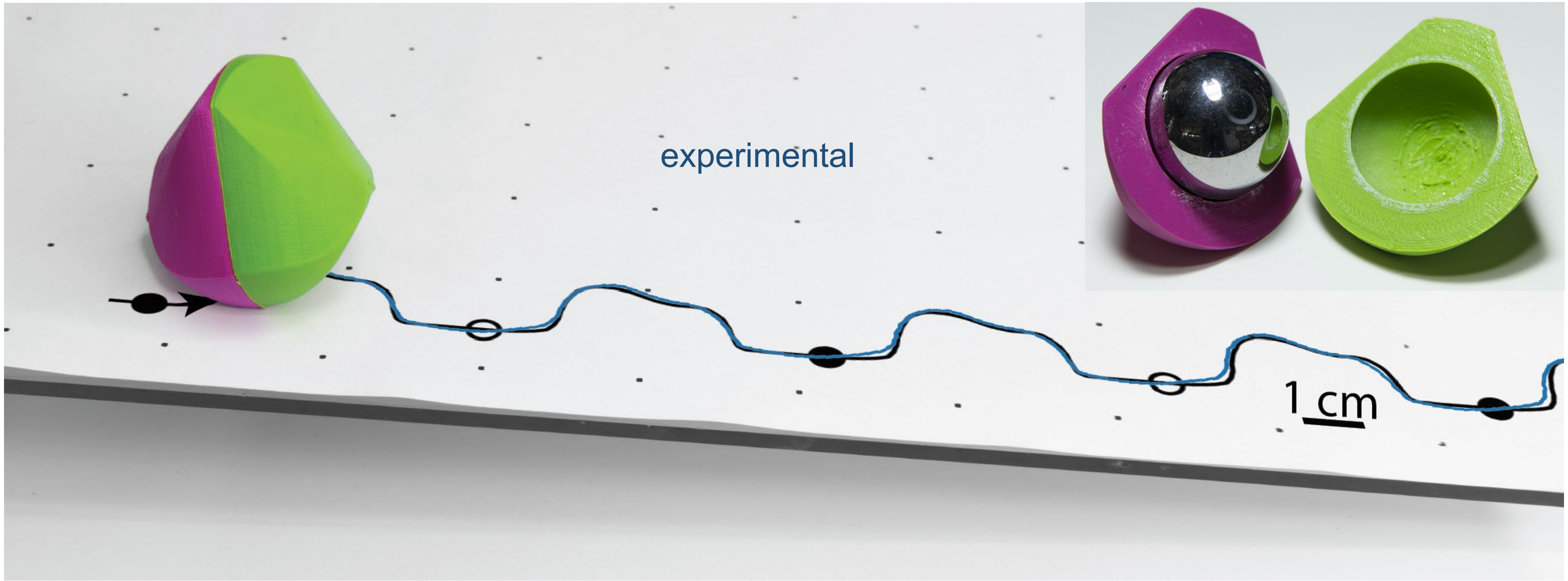


radius→∞ ⟶ small radius

Property π is obeyed by a rich variety of paths ('almost any')

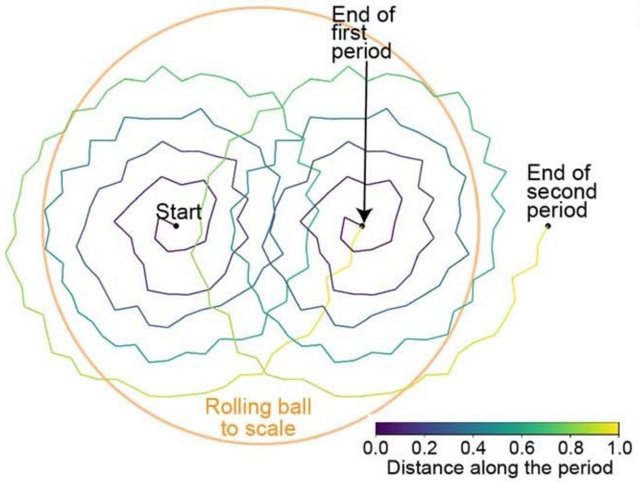


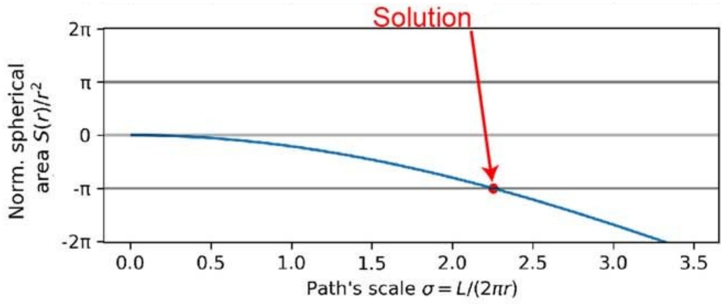


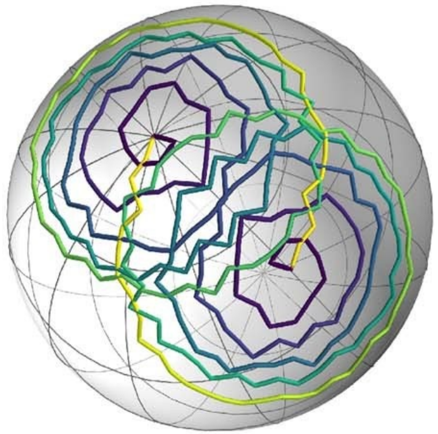


… also for convoluted self-intersecting paths

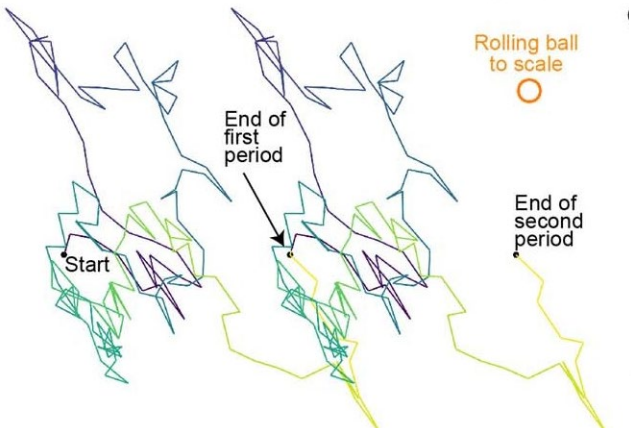
spiral + noise

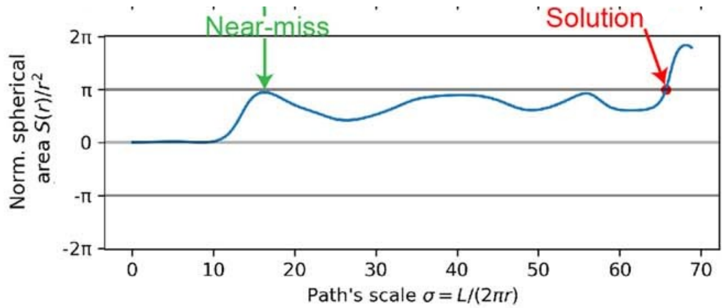


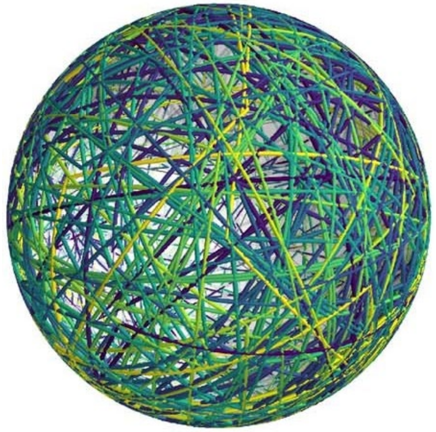




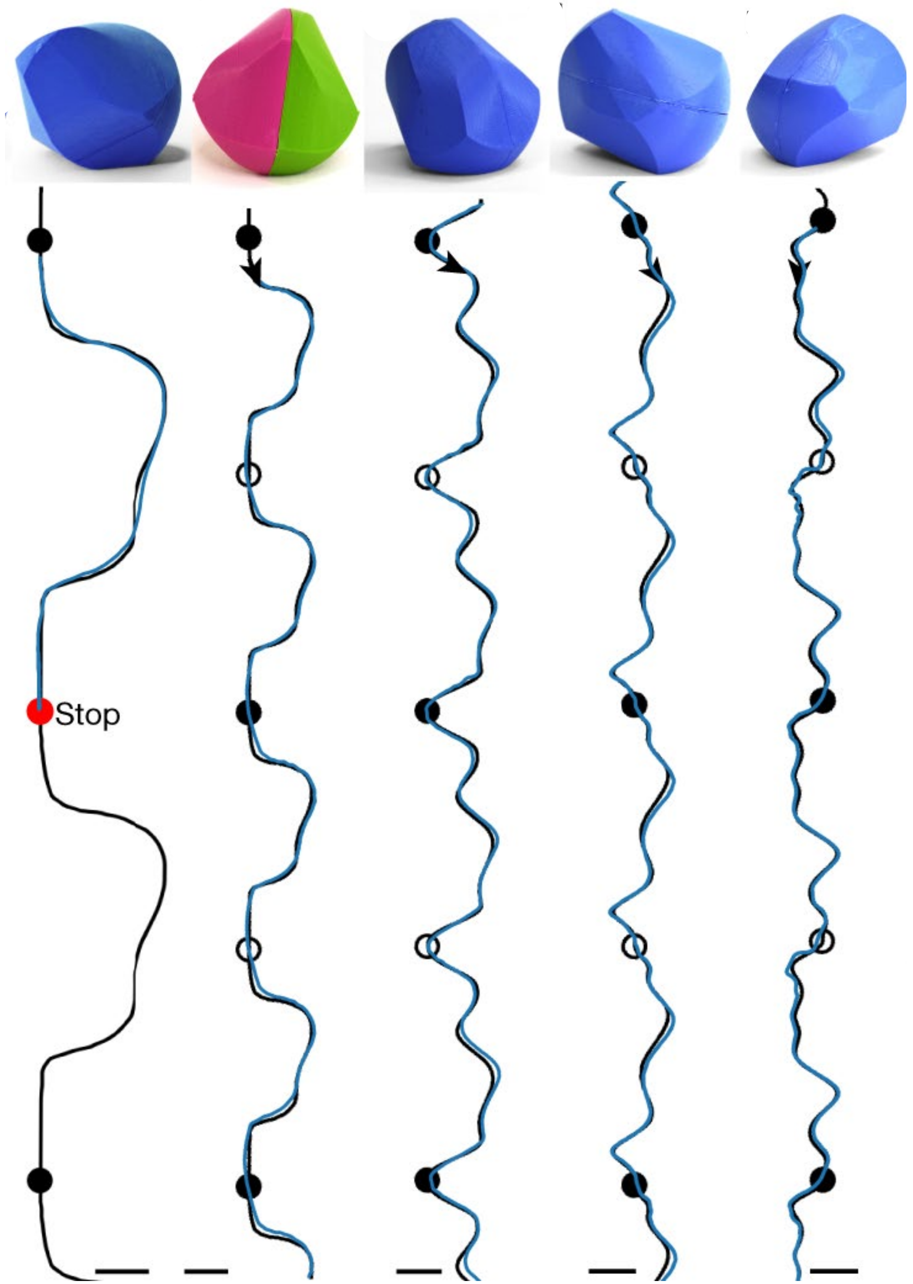
2D random walk





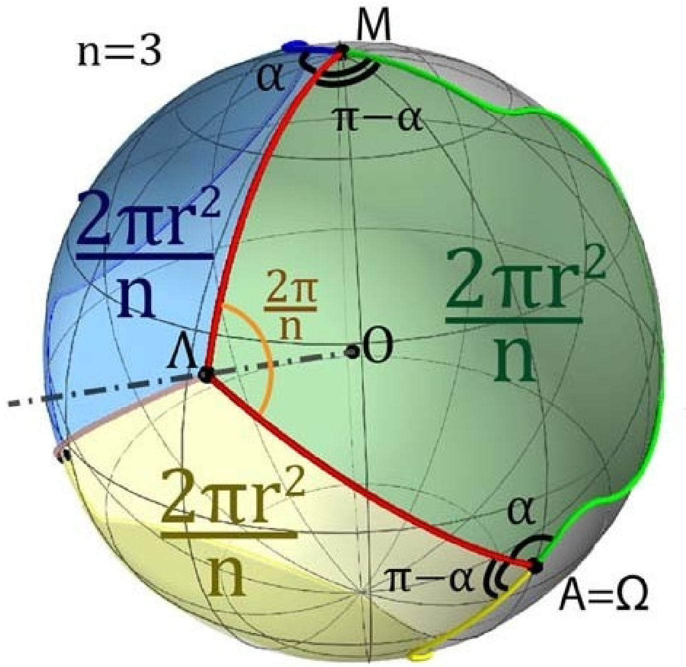


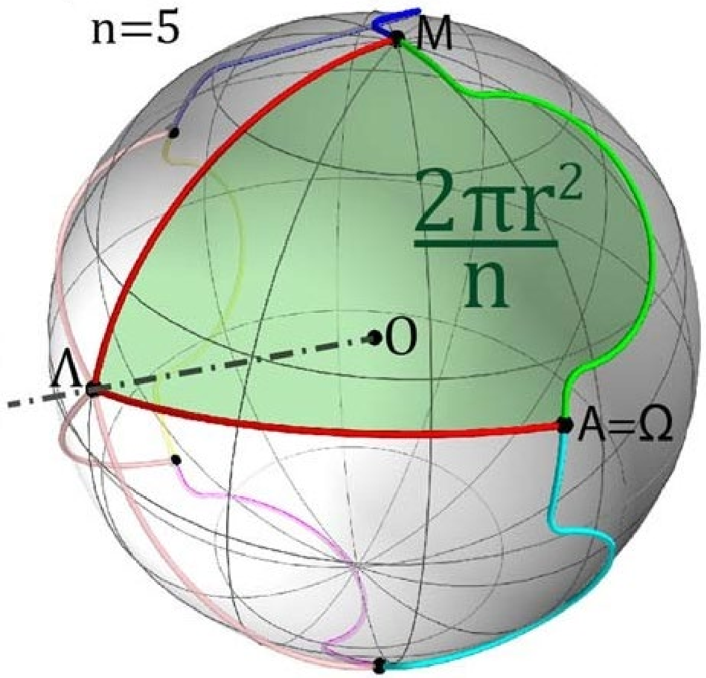
please ask questions during the talk

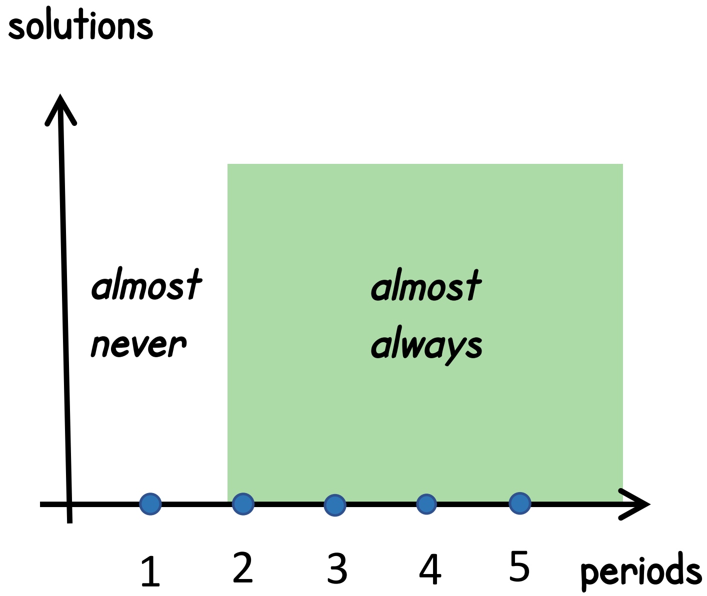


Build your own trajectoid at
https://colab.research.google.com/drive/1XZ7Lf6pZu6nzEuqt_dUCHormeSbCCMIP

n-period trajectoids almost always exist for any $n \geq 2$



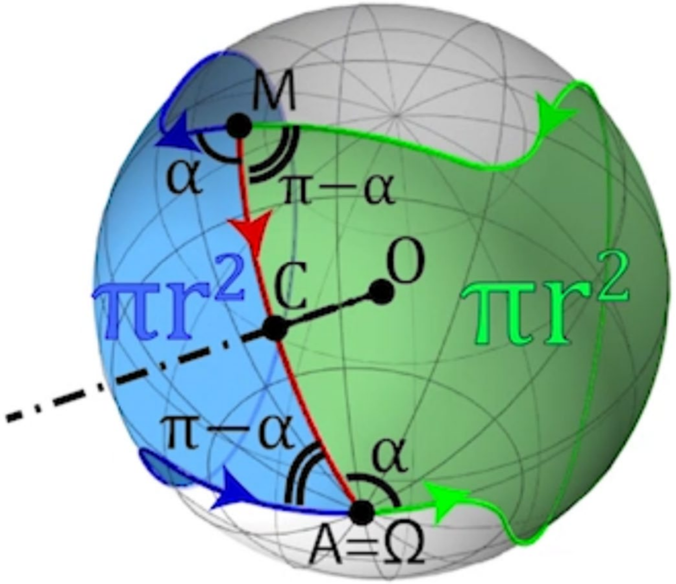




Construction using the $2\pi/n$ property → Trajectoids have C_n symmetry.

Please ask questions during the talk

TPT is a simple general feature of the rotation group SO(3)



For a general product of coplanar rotations

$$R(\mathbf{n}_1,\theta_1)R(\mathbf{n}_2,\theta_2)\ldots R(\mathbf{n}_m,\theta_m)$$

there is $r>0$

$$\left[R(\mathbf{n}_1,\tfrac{1}{r}\theta_1)R(\mathbf{n}_2,\tfrac{1}{r}\theta_2)\ldots R(\mathbf{n}_m,\tfrac{1}{r}\theta_m)\right]^2=1$$

Conjecture and experimental fact: 2-period trajectoids exist for almost any path.

Theorem: for any path, an m-period trajectoid exists for a finite m.

Therefore, TPT can be translated to other physical systems governed by SO(3)

Rotation

$$R_{A\Omega} = \mathcal{T}\left[\exp\left(\frac{1}{r}\int_{A}^{\Omega} dt\, \mathbf{n}(t)\cdot\vec{\mathbf{L}}\right)\right] \quad \rightarrow \quad U_{A\Omega} = \mathcal{T}\left[\exp\left(-\frac{i}{\hbar}\int_{A}^{\Omega} dt\, H(t)\right)\right]$$

Time evolution

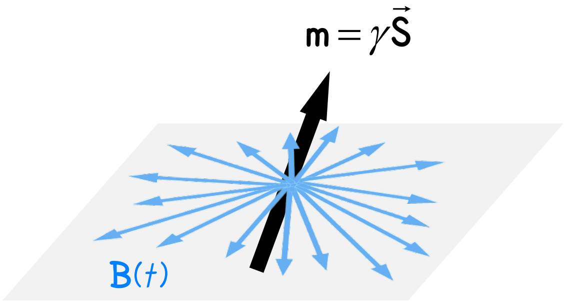
with time-dependent $H(t) \sim \mathbf{n}(t)\cdot\vec{\mathbf{L}}$

'Almost any' planar field pulse n(t)

- scaled by an appropriate r > 0 and applied twice -

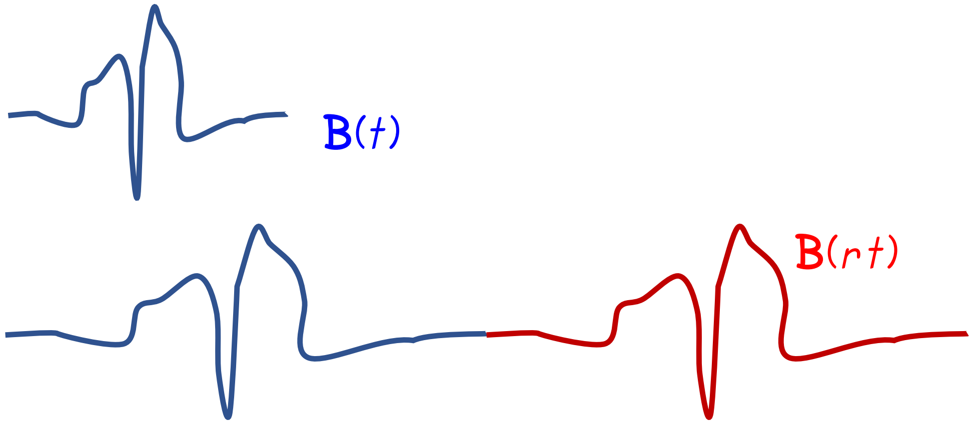
will return the system exactly to its original state.

Planar magnetic pulses can be rescaled such that the doubled pulse returns the dipole to its initial orientation



$$H(t) = -\mathbf{B}(t)\cdot\gamma\vec{\mathbf{S}} \qquad (\gamma = \text{gyromagentic ratio})$$

$$-\frac{i}{\hbar}\int_A^\Omega dt\, H(t) = \int_A^\Omega dt\,\gamma\mathbf{B}(t)\cdot\vec{\mathbf{L}} = \int_A^\Omega [dt\,\gamma B(t)]\,\hat{\mathbf{B}}(t)\cdot\vec{\mathbf{L}} = \int_A^\Omega \frac{dt}{r}\mathbf{n}(t)\cdot\vec{\mathbf{L}}$$

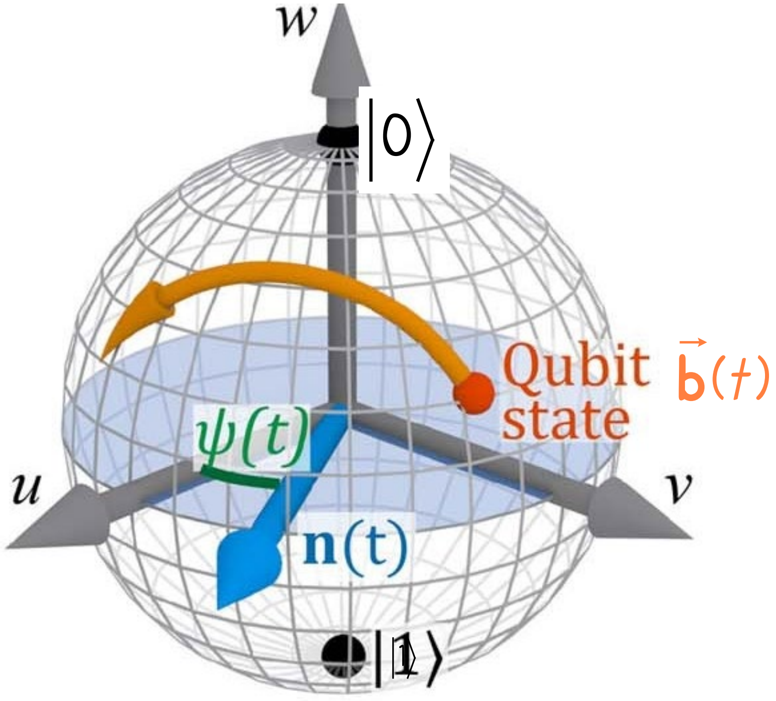


Scaling by either

the field magnitude: $B(t) \to rB(t)$

or rescaling time: $B(t) \to B(r\,t)$

Rotation of a qubit in an electric field can be rescaled to return to the original state by applying TPT

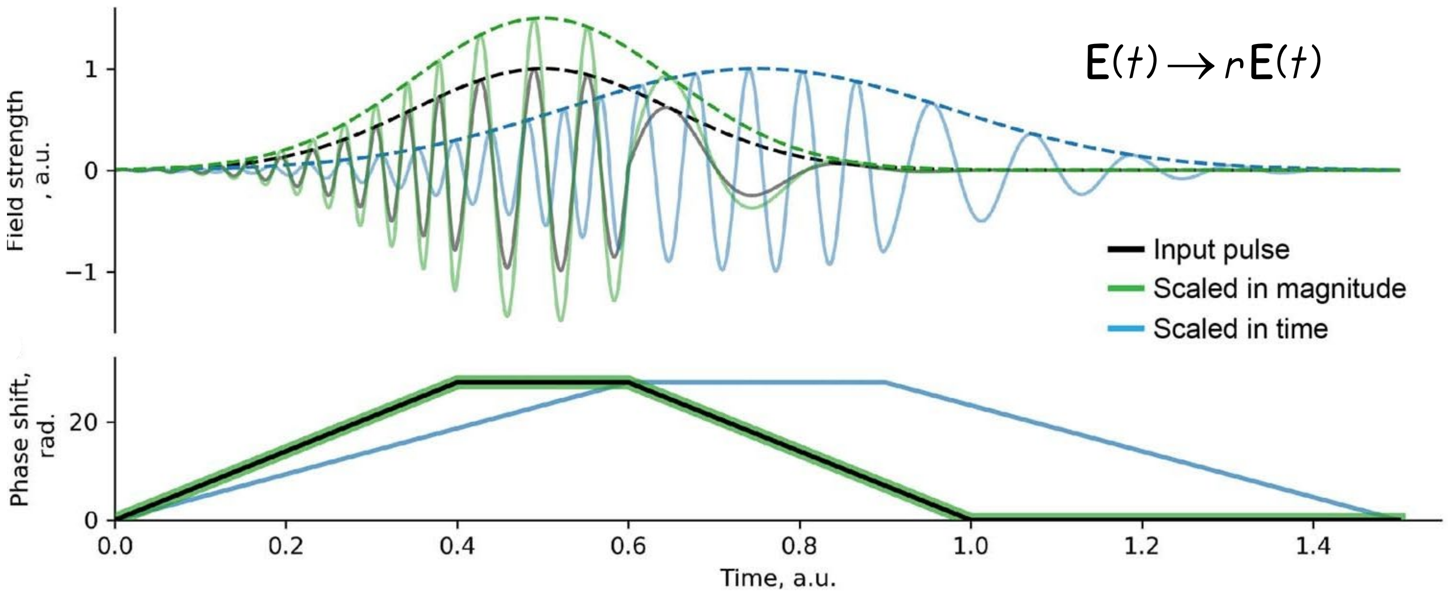


$$|\mathbf{b}(t)\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

$$\Leftrightarrow \vec{\mathbf{b}}(t) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

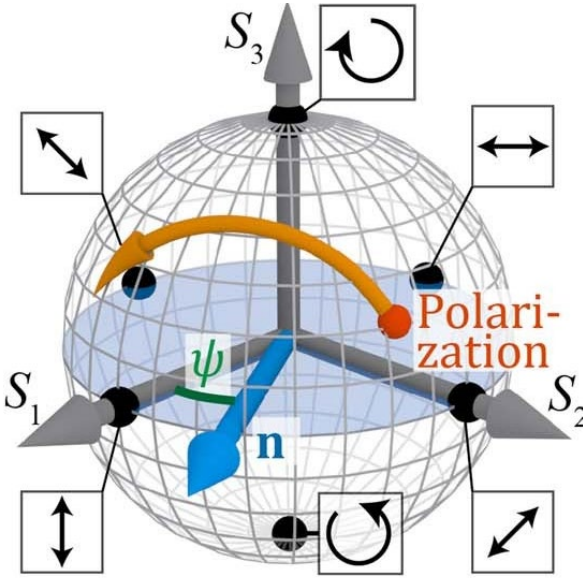
Bloch vector $\vec{\mathbf{b}}$ in electric field $E(t)\cos\left(\omega_0 t - \psi(t)\right)$ (resonance $\omega_0 = \frac{W}{\hbar}$)

turns at angular velocity $\mathbf{n}(t) = -\frac{\mu_{ab}}{\hbar}E(t)\left(\cos\psi(t), \sin\psi(t), 0\right)$.



$E(t) \rightarrow E(rt), \quad \psi(t) \rightarrow \psi(rt)$

Analogous application applies to NMR pulses with planar B(t)

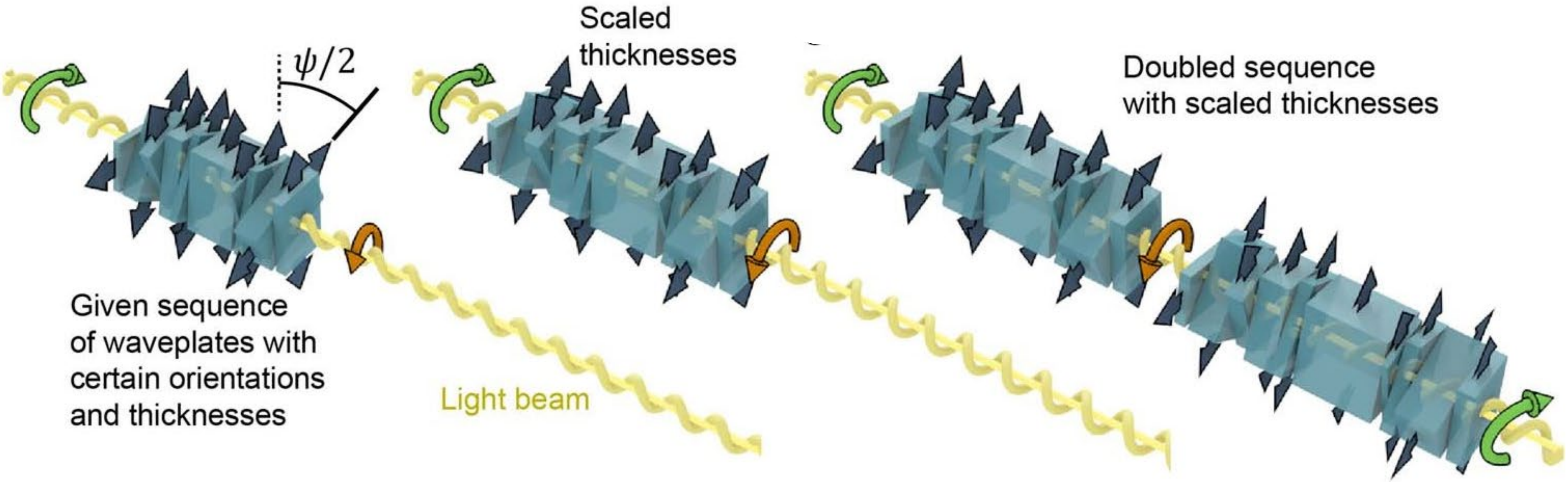


Light passing through 'almost any' stack of waveplates returns to original polarization if the stack is rescaled and doubled by TPT

Polarization vector of light passing a thin uniaxial waveplate rotates by an angle δt_i plate thickness around axis $\mathrm{n}(t) = \big(\cos\psi(t), \sin\psi(t), 0\big)$.

($\frac{1}{2}\psi =$ waveplate optical axis)

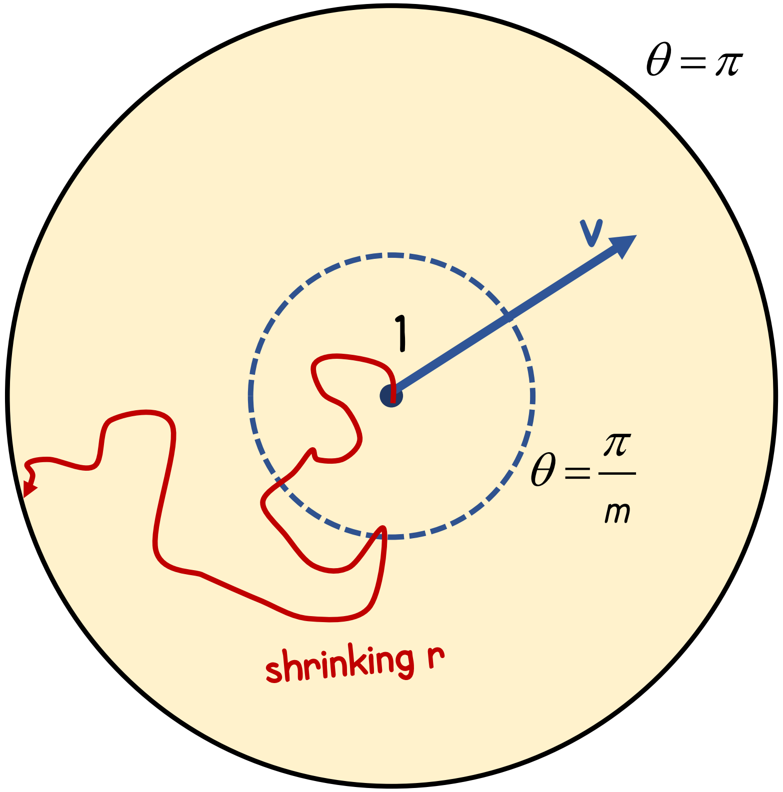
$$
\begin{aligned}
S_0 &= I = E_x^2 + E_y^2 \\
S_1 &= Q = E_x^2 - E_y^2 \\
S_2 &= U = 2E_xE_y \cos\delta \\
S_3 &= V = 2E_xE_y \sin\delta
\end{aligned}
$$

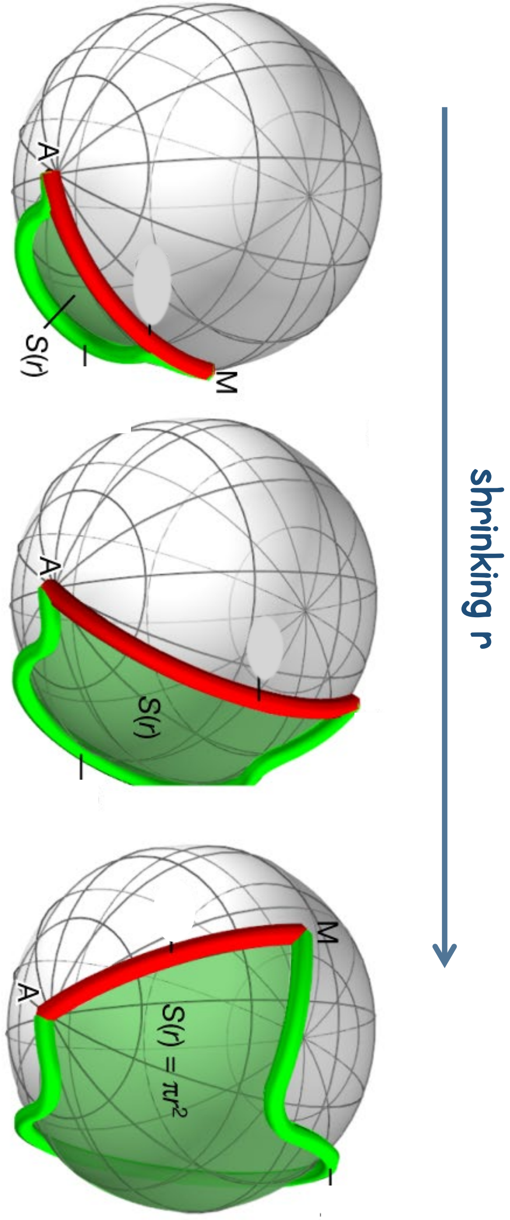


Why two-period trajectoids are generic?

Space of all rotations in SO(3)

$\vec{\mathbf{v}} \sim \mathbf{n}, \quad |\vec{\mathbf{v}}| = \theta$





TRAJECTOIDS for almost any path exist

- a mathematical property of the rotation group
- with some applications in SO(3) systems

- Why 2-period paths are dense?
- More applications?
- Generalization to higher dimensions?

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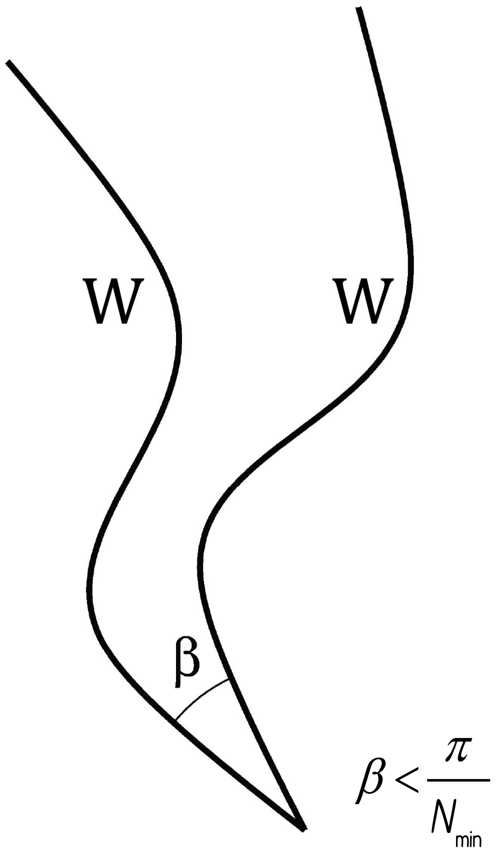
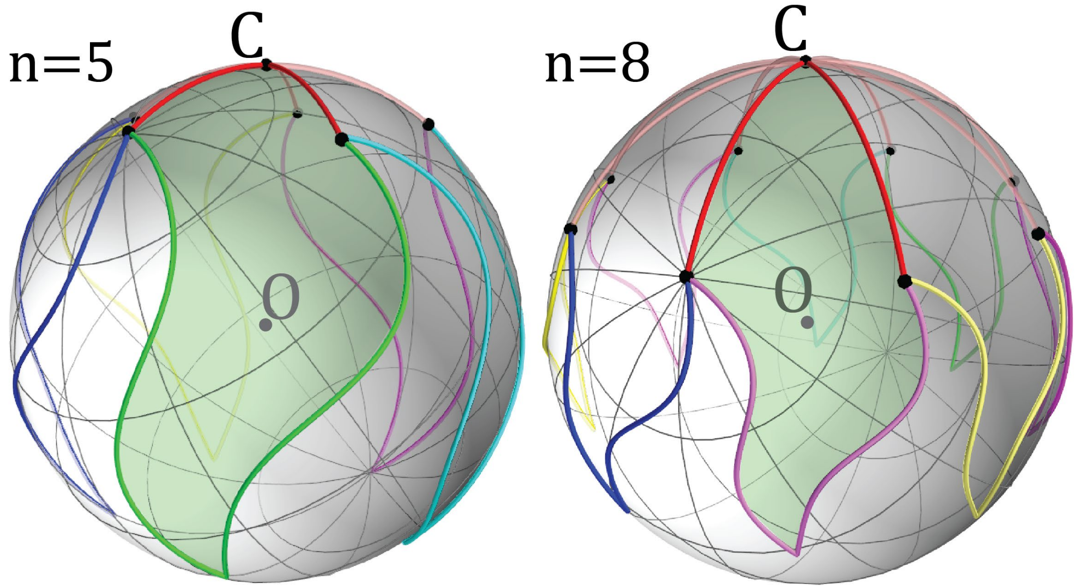
Eckmann et al. Tumbling downhill along a given curve. *Amer Math Soc Notices* - in press.

Sobolev et al. Solid-body trajectoids shaped to roll along desired pathways. *Nature* 2023.

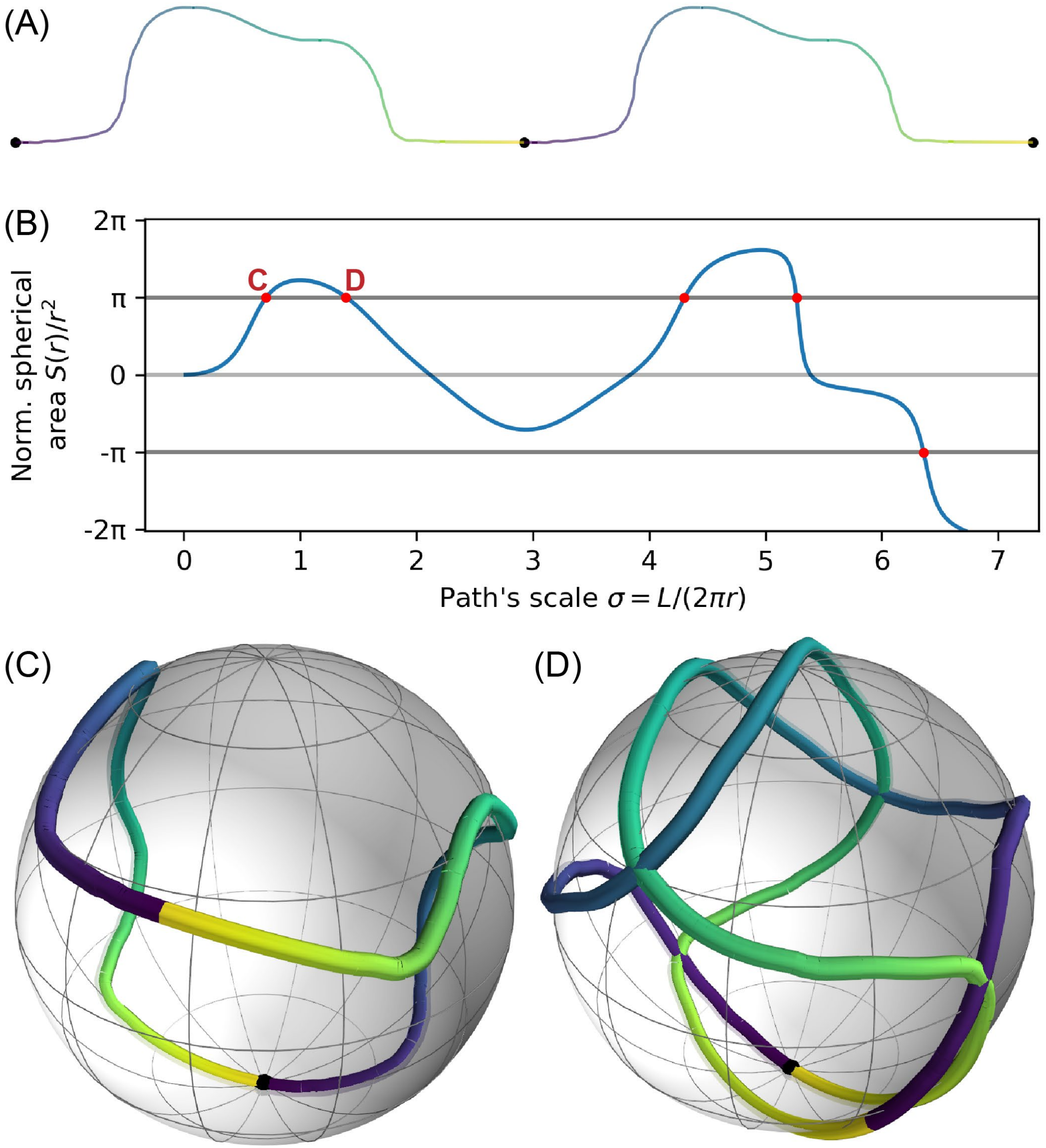
There are rare symmetric paths for which trajectoids exist only for $N > N_{min}$





NMR…

(2) In the NMR context, a strong static magnetic field B_0 along the *z* axis causes precession of the dipole moment around that axis with Larmor frequency $\omega_0 = \gamma B_0$, and a pulse of magnetic field $B_1(t) = B_1(t)(\hat{x}\cos(\omega_0 t - \psi(t)) + \hat{y}\sin(\omega_0 t - \psi(t)))$ rotating in the *xy* plane is applied. For example, a linear increase of the phase shift $\psi(t) = t\Delta$ will occur when the pulse is detuned slightly off-resonance[25,26]: that is, it oscillates with frequency $(\omega_0 - \Delta)$ instead of ω_0. If the phase shift $\psi(t)$ and magnitude $B_1(t)$ change slowly with time t, it is common to use the reference frame that rotates together with Larmor precession and formulate optical Bloch equations in a rotating wave approximation[26,27], where state vector $\rho(t)$ is the normalized magnetic dipole moment in the rotating frame, and its instantaneous angular velocity $\mathbf{n}(t) = -\gamma B_1(t)[\hat{x}\cos\psi(t) + \hat{y}\sin\psi(t)]$ belongs to the *xy* plane. For the application of the TPT theorem, see the discussion after (3).



(A)
(B)
2π
π
0
-π
-2π
C
D
Norm. spherical area $S(r)/r^2$
0
1
2
3
4
5
6
7
Path's scale $\sigma = L/(2\pi r)$
(C)
(D)