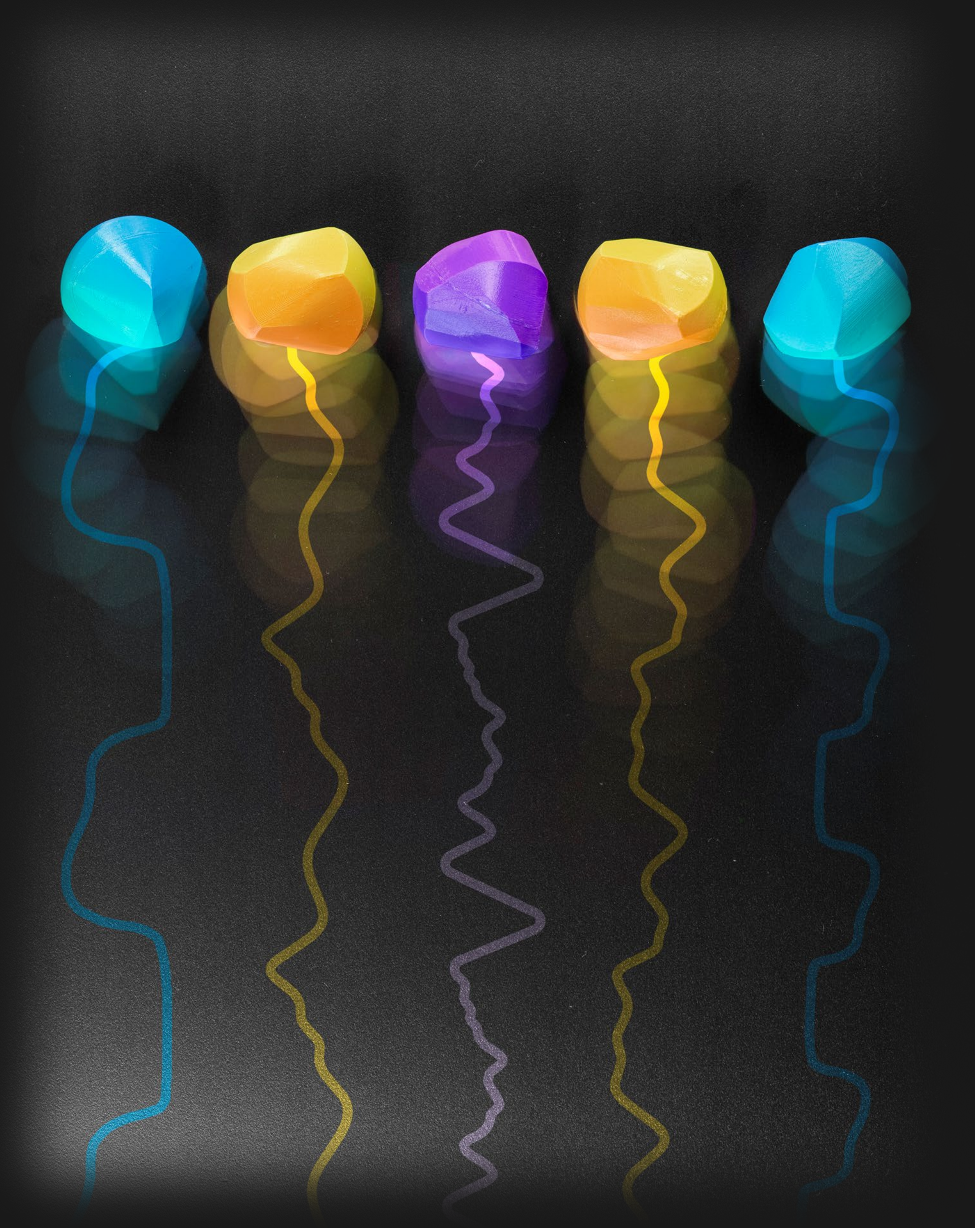
Courant – applied math seminar

Double it, scale it, and your walk in rotation space comes right back

Yaroslav SOBOLEV • Jean-Pierre ECKMANN

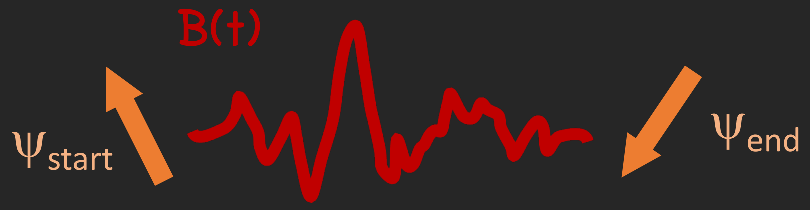


UNIST
ULSAN NATIONAL INSTITUTE OF SCIENCE AND TECHNOLOGY
2 0 0 9

Sobolev et al. Nature 2023; **Eckmann** et al. AMS Notices 2024; **Eckmann** et al. arXiv:2502.14367

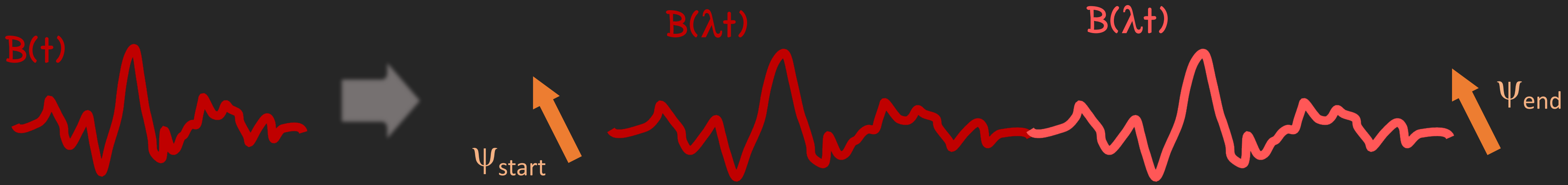
Many physical systems—like dipoles, spins, and qubits—are governed by SU(2) or SO(3)

Consider a spin or magnetic dipole $\psi(t)$ evolving under the influence of a complicated magnetic field pulse B(t).

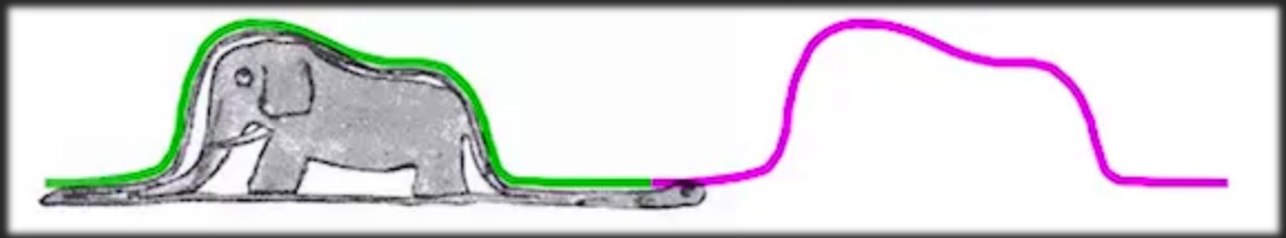


Question: Can we make the pulse rewind the system to its original state?

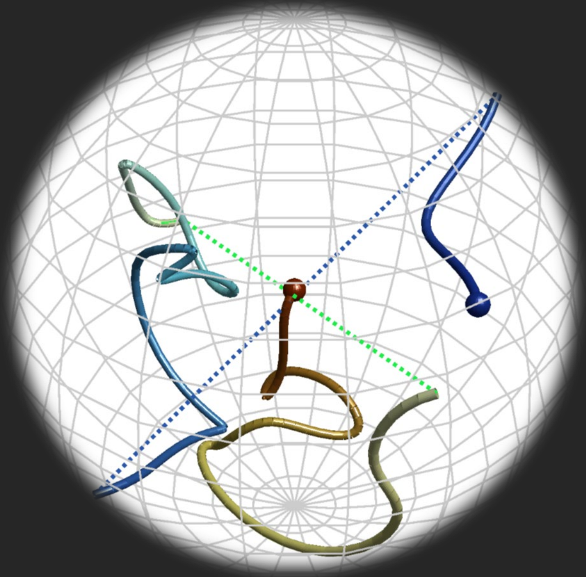
Result: YES—scale the pulse and apply it twice (or more), and the system returns to its original state



Outline: Walks in rotation spaces return home when doubled and scaled



1. The result.
2. Where this begins: rolling along infinite wiggly paths on the plane.
3. Underlying physical intuition: Walks in Lie groups and random rotation ensembles
4. Mathematical framing: Diophantine equations and Minkowski's theorem
5. Conclusion: Is this trick useful for anything?



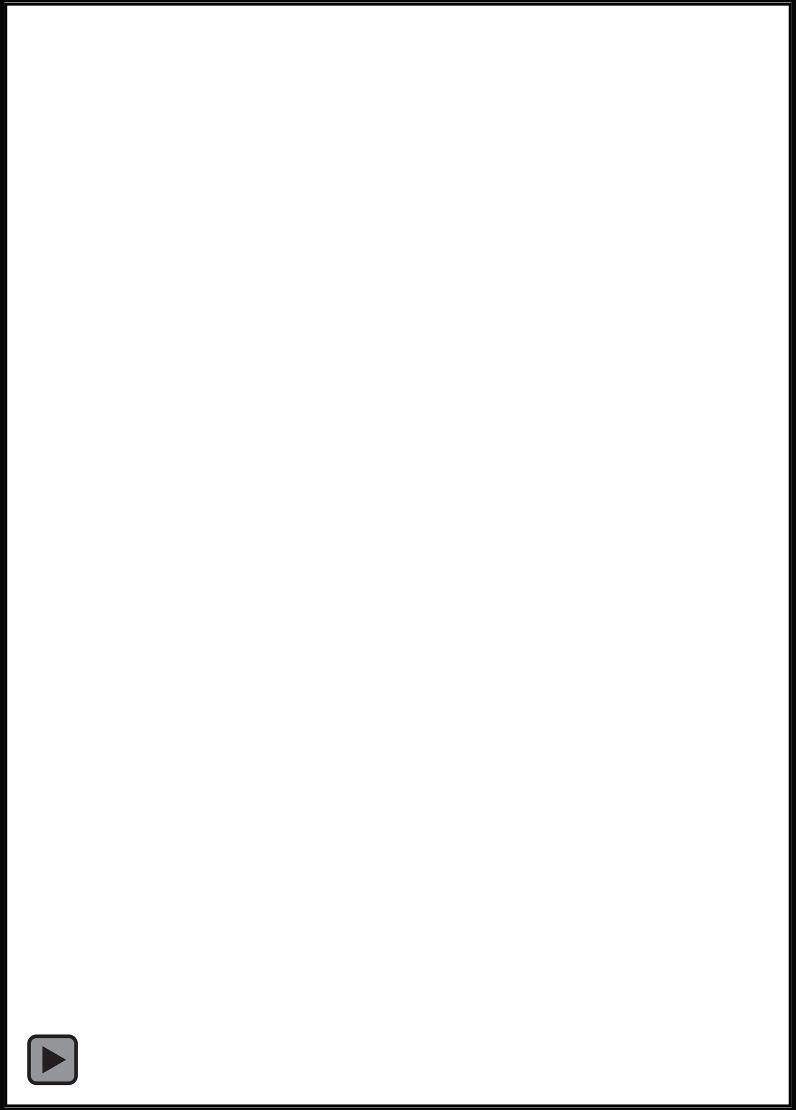
Please ask questions during the talk

2. ROLLING

Minnesota Fats and Eddie Felson
The Hustler (1961)

Ben-Hur (1959)

Artists, engineers, and mathematicians discovered more exotically shaped solid rollers



But let's look at the more general *inverse* problem:

Is it possible to carve a shape that

rolls along <u>any</u> given periodic path?

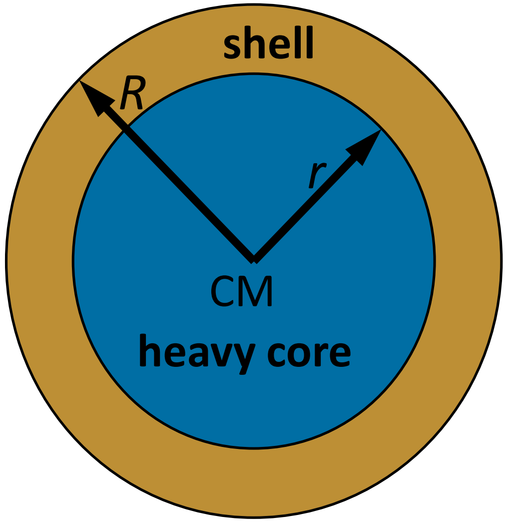
Sobolev et al. Nature 2023; **Eckmann** et al. AMS Notices 2024

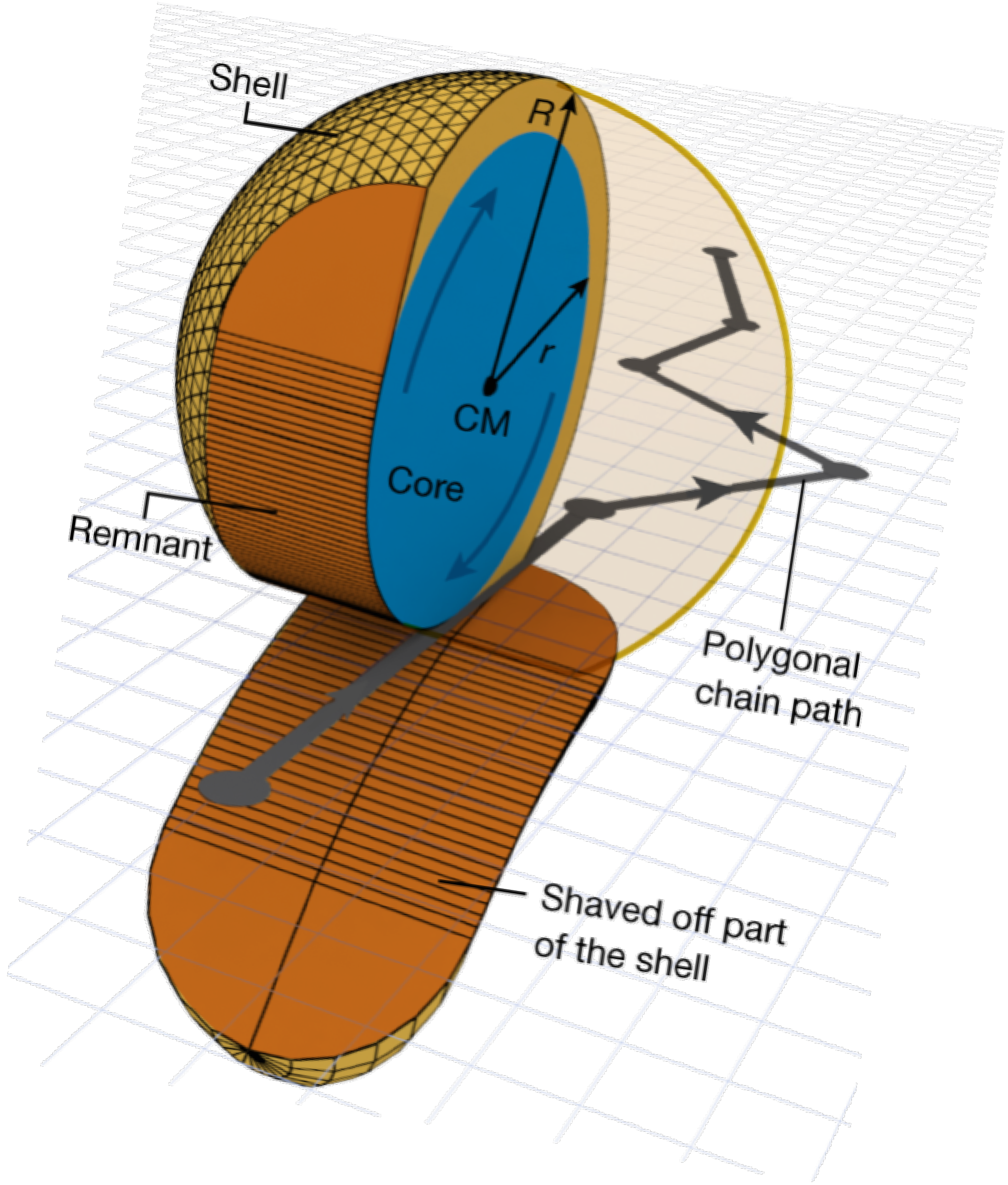
THE BASIC SETTING

- Draw a curve T on a plane.
- Repeat the curve to construct an infinite periodic path, T_∞
- Tilt a bit the plane slightly (or roll by hand)

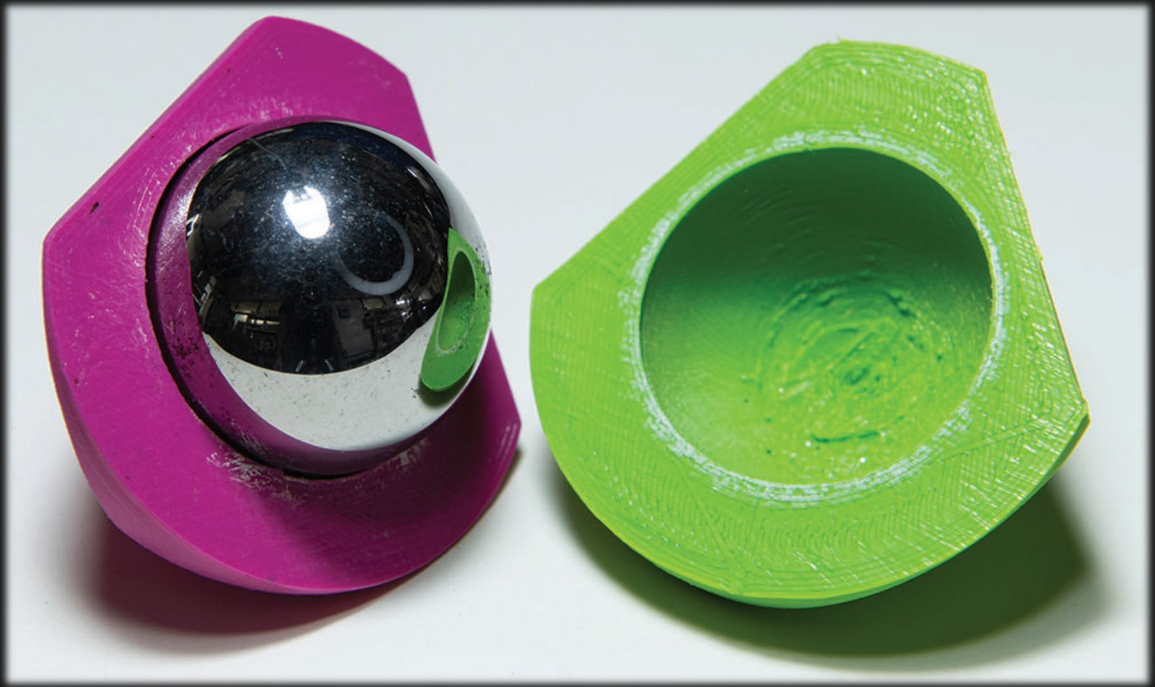
Now, sculpt a solid body - a 'trajectoid' - that rolls *perfectly* along T_∞ without slipping or pivoting (spinning).

The LOCAL problem of rolling along a general path is solved by a "shaving" construction



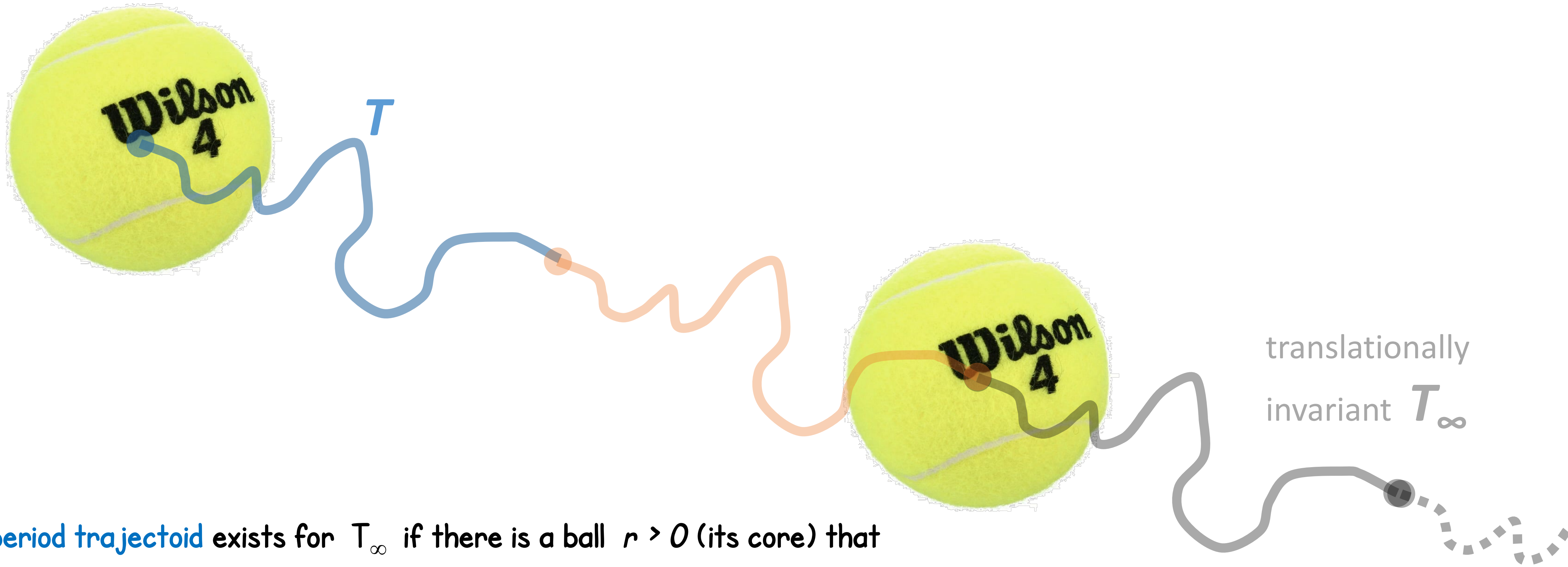


These trajectoids are produced in a simple 3D printer



To solve the GLOBAL problem:

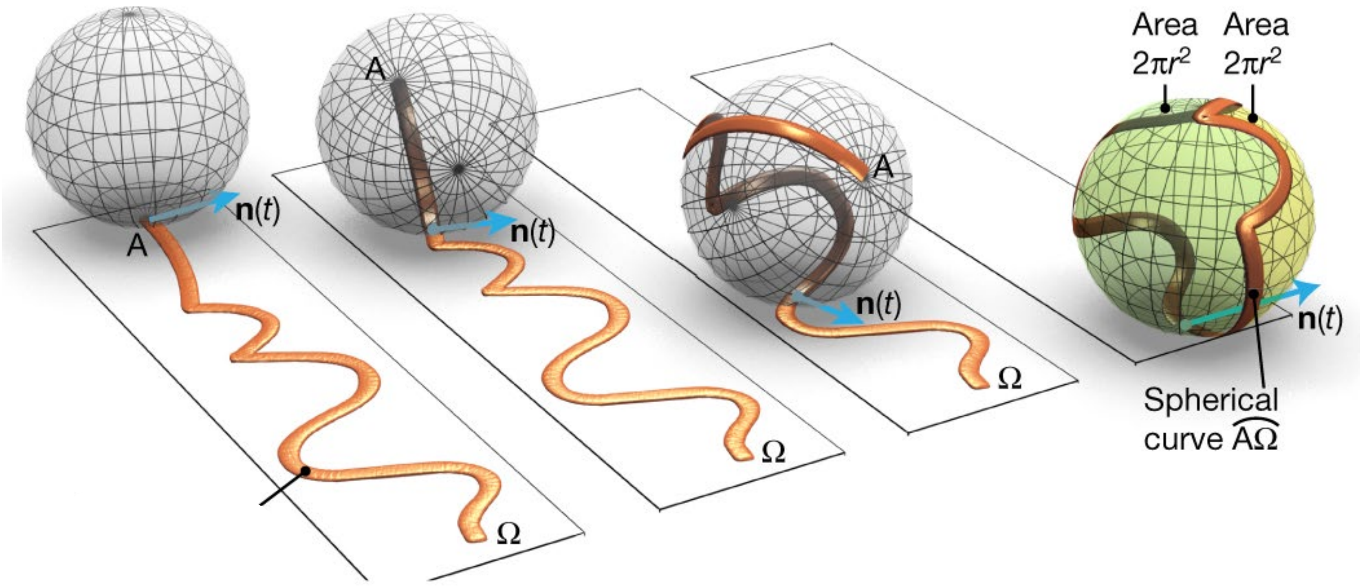
A trajectoid must periodically regain its initial 3D orientation



n-period trajectoid exists for T_∞ if there is a ball $r > 0$ (its core) that

after rolling along n periods regains its original orientation (without slipping or pivoting)

The trajectoid problem naturally reduces to tracking how the core rotates as it rolls



Path $\mathbf{A\Omega}$ defined by normal $\mathbf{n}(\boldsymbol{t})$

Each segment $\boldsymbol{\delta t}$ induces a rotation by $\boldsymbol{\delta t/r}$ around n(t)

$$R_i = \exp\left(\frac{\delta t}{r}\left(\mathbf{n}(t_i)\cdot\vec{\mathbf{L}}\right)\right)$$

Regaining orientation → overall rotation is the identity:

$$R_{A\Omega} = \prod_i R_i = \mathcal{T}\left[\exp\left(\frac{1}{r}\int_A^\Omega dt\,\mathbf{n}(t)\cdot\vec{\mathbf{L}}\right)\right] = 1$$

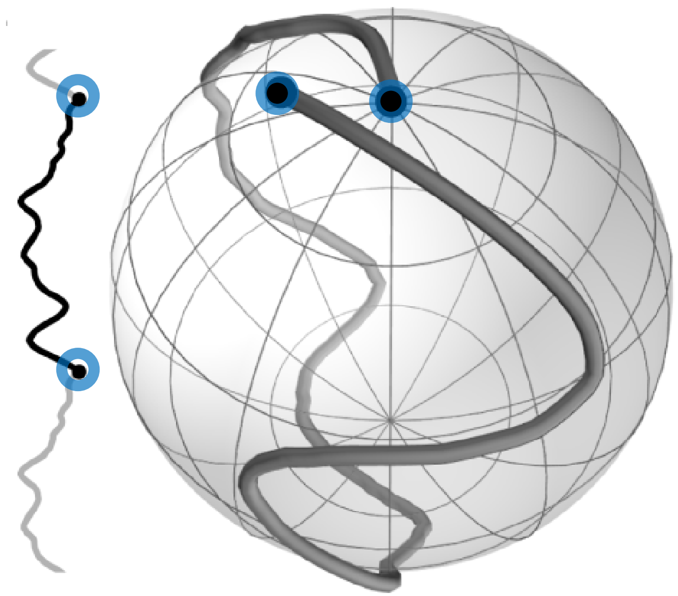
rotation group generators

$$L_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad L_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad L_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

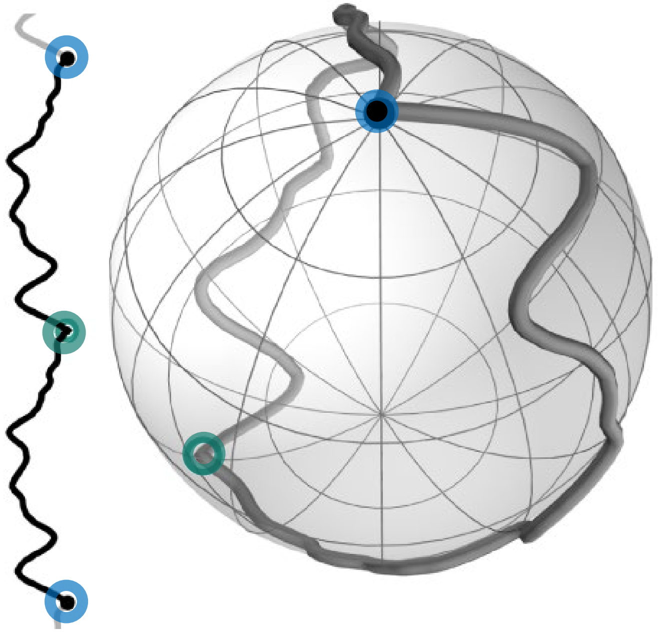
please ask questions during the talk

Experimental fact

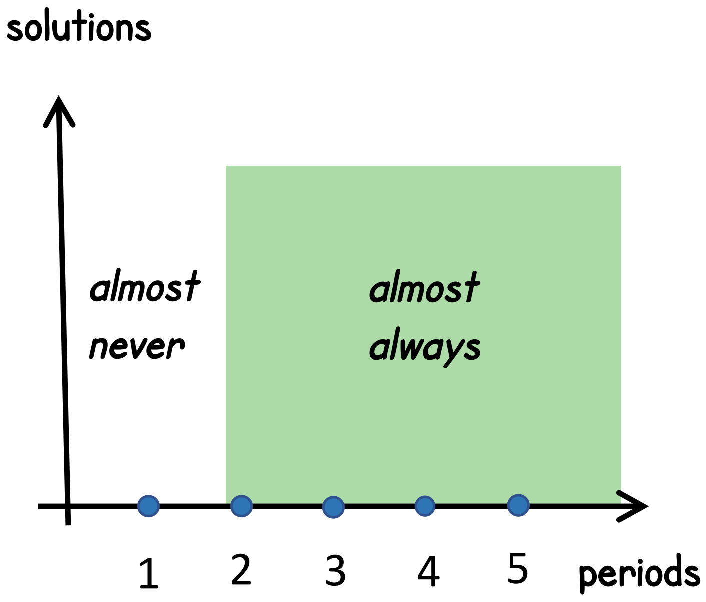
The difficulty of finding a trajectoid depends sharply on the number of periods



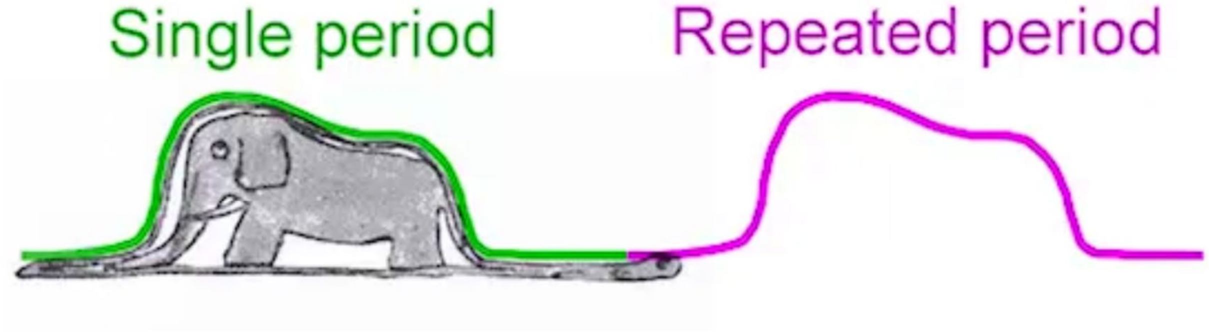
1-period trajectories
almost never exist

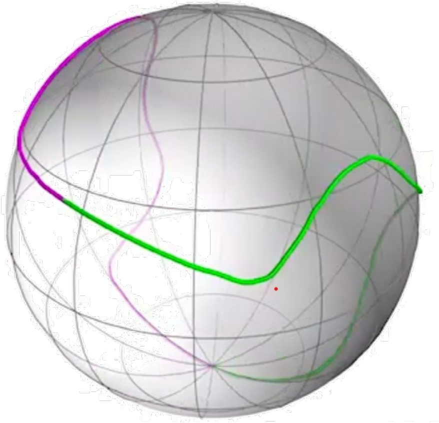


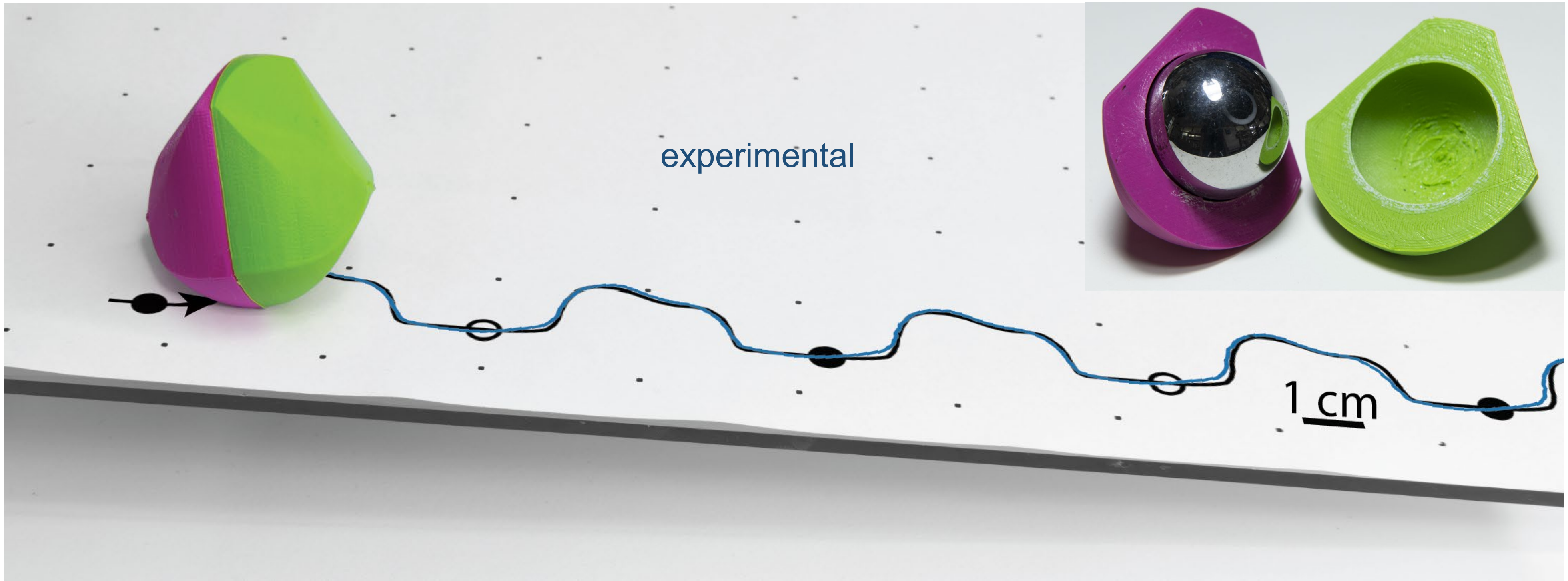
2-period trajectories
almost always exist

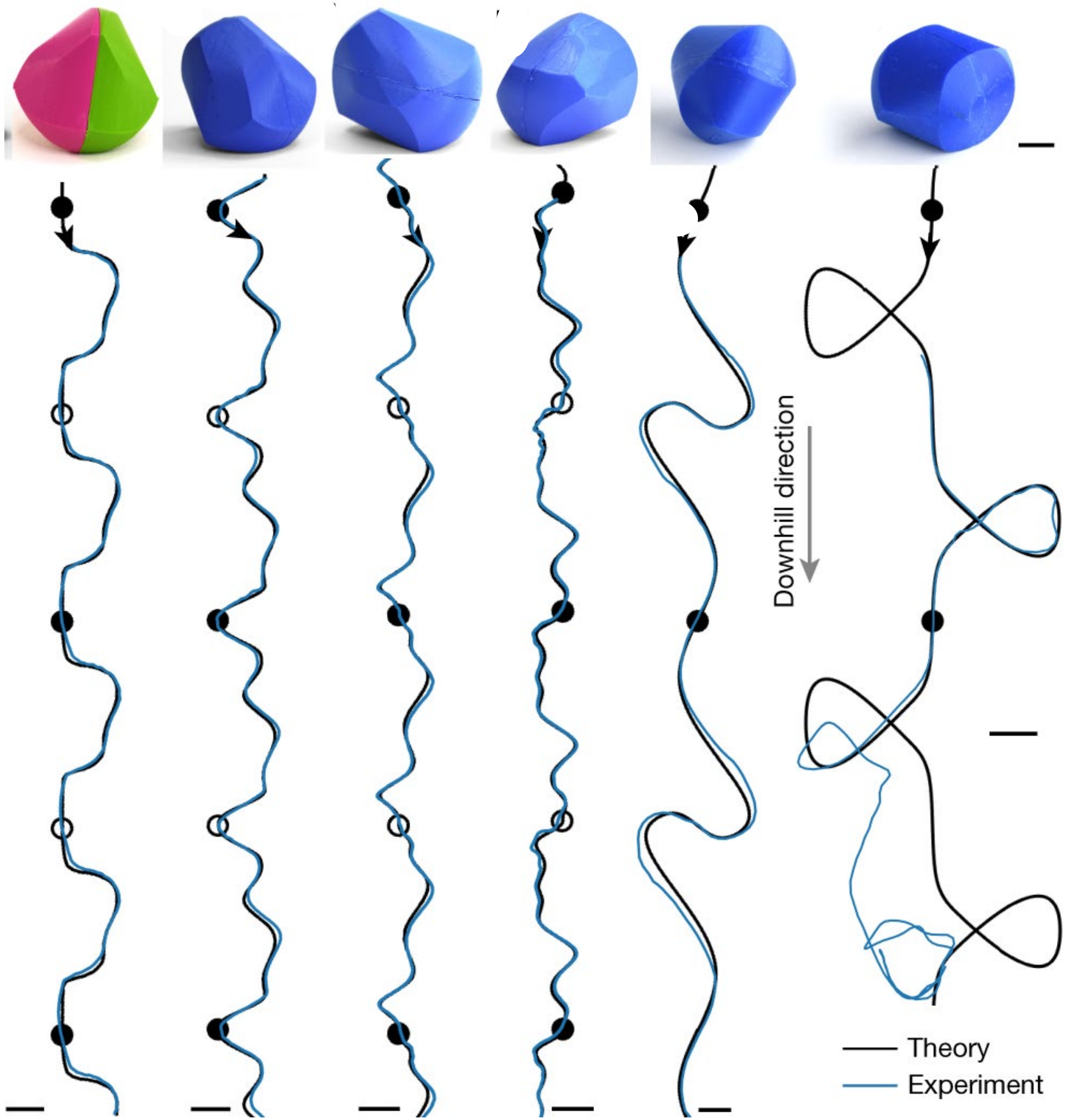


Almost any path allows trajectoids with multiple periods—two or more.





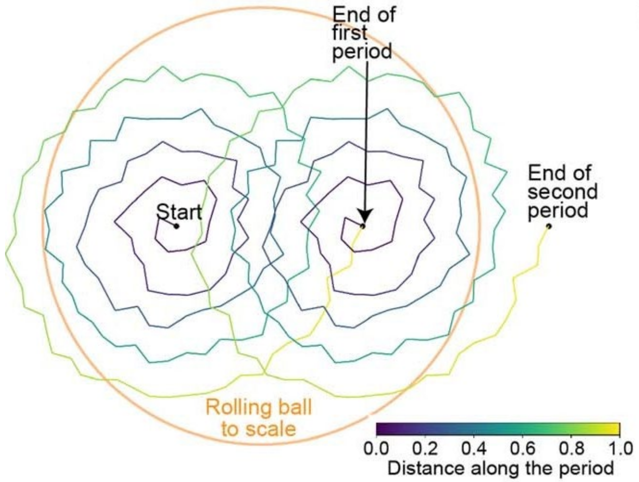


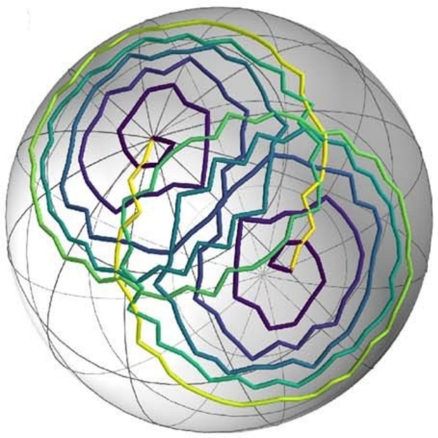


Build your own trajectoid at
https://colab.research.google.com/drive/1XZ7Lf6pZu6nzEuqt_dUCHormeSbCCMIP

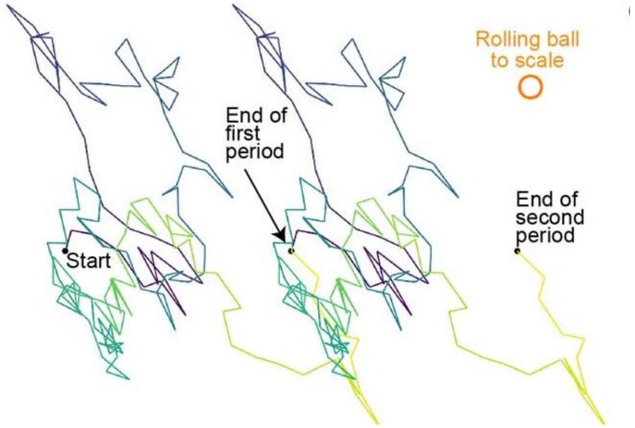
This remains true also for extremely convoluted self-intersecting paths

spiral + noise

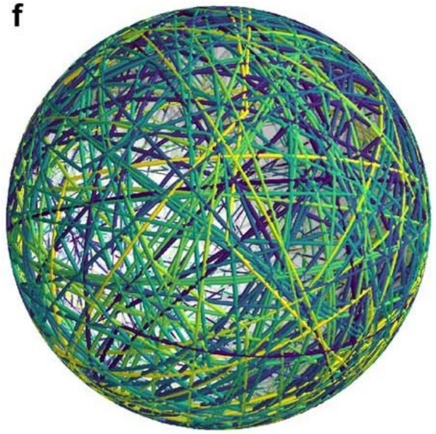




2D random walk



f



3. Walks in Lie groups:

To grasp this experimental conjecture, forget rolling and study the rotation group directly.

Eckmann et al. arXiv:2502.14367

Rotation product are walks in the 3D manifold of SO(3)

Rotation $R \in SO(3)$ is specified by 3 parameters,

e.g. an axis n and rotation angle $\omega \in [0, \pi]$

$$\boxed{R(\mathrm{n}, \omega) = e^{\omega\left(\mathrm{n}\cdot\vec{\mathrm{L}}\right)}}$$

rotation group generators

$$L_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad L_y = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad L_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In terms of the field: the axis is $\mathrm{n} = \hat{\mathrm{B}}$ and the angle $\omega \sim |\mathrm{B}|$

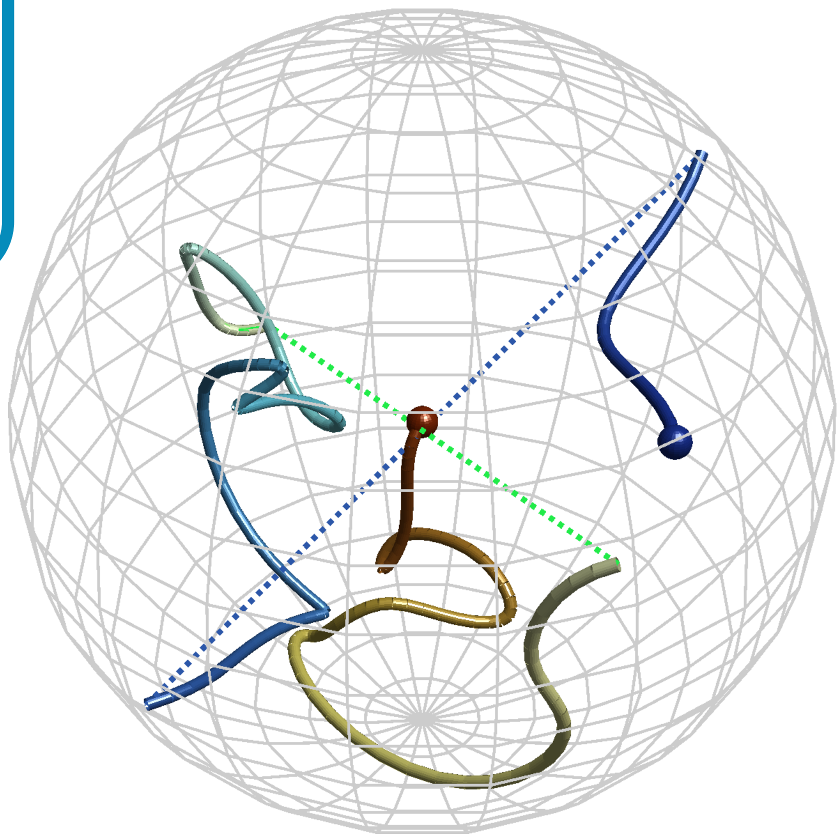
$H(t) = -\mathrm{B}(t)\cdot \gamma \vec{\mathrm{S}}$, $\quad \vec{\mathrm{S}} = i\hbar \vec{\mathrm{L}}$ $\quad$ (γ = gyromagnetic ratio)

$$U(t) = e^{-\frac{i}{\hbar} dt\, \mathrm{H}(t)} = e^{dt\, \gamma \mathrm{B}(t)\cdot\vec{\mathrm{L}}} = e^{\left(dt\, \gamma |\mathrm{B}(t)|\right)\left(\hat{\mathrm{B}}(t)\cdot\vec{\mathrm{L}}\right)} = e^{d\omega\left(\mathrm{n}(t)\cdot\vec{\mathrm{L}}\right)} = R(\mathrm{n}, d\omega)$$

Series of rotations can be visualized as walks inside a ball (actually $\mathbb{RP}^3$)

Map a rotation $R(n,\omega) \mapsto r = n\omega$, a point in a ball of radius π :

- Identity $R(n,0)=1$ is the center
- π-rotations are surface with antipodes identified, $R(n,\pi)=R(n,-\pi)$.



2:1 Mapping from $SU(2) \cong S^3$ to $SO(3) \cong \mathbb{RP}^3$:

$$R(n,\omega) = e^{\omega(n\cdot\vec{L})} \mapsto \pm U, \quad U\left(n,\tfrac{1}{2}\omega\right) = e^{\frac{i}{2}\omega(n\cdot\vec{\sigma})} = I\cos\tfrac{\omega}{2} + \left(n\cdot\vec{\sigma}\right)\sin\tfrac{\omega}{2}$$

or to unit quaternions: $\quad R(n,\omega) \mapsto \pm q, \quad q = \cos\tfrac{\omega}{2} + \sin\tfrac{\omega}{2} n$

with the Pauli matrices $\vec{\sigma} = \left(\sigma_x, \sigma_y, \sigma_z\right)$

Altmann, Rotations, quaternions, and double groups, 2005

In terms of walks in the SO(3) ball:

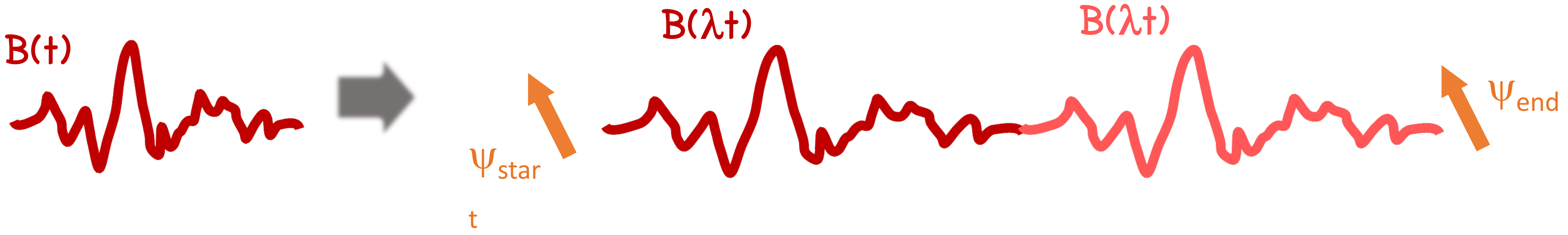
Q: Given a general product of rotations, the walk $W = \prod_{i=1}^{N} R_i = R_N \ldots R_2 R_1$,

Find a scaling factor $\lambda > 0$ of all matrices $\left[R(n,\omega)\right]^{\lambda} = R(n, \lambda\omega)$,

such that $\quad W(\lambda) = \prod_{i=1}^{N} R_i^{\lambda} = 1$? (returning "home")

RESULT: The problem is almost always solvable if the walk is repeated more than once.

$\left[W(\lambda)\right]^{m} = 1$, for any integer $m \geq 2$, but almost never for $m = 1$.



3. Walks in Lie groups and random rotation ensembles

One may tackle the problem probabilistically:

Test multiple values of λ and see if $W(\lambda) \to 1$.

Eckmann et al. arXiv:2502.14367

We therefore need to think about the ensemble of random rotations

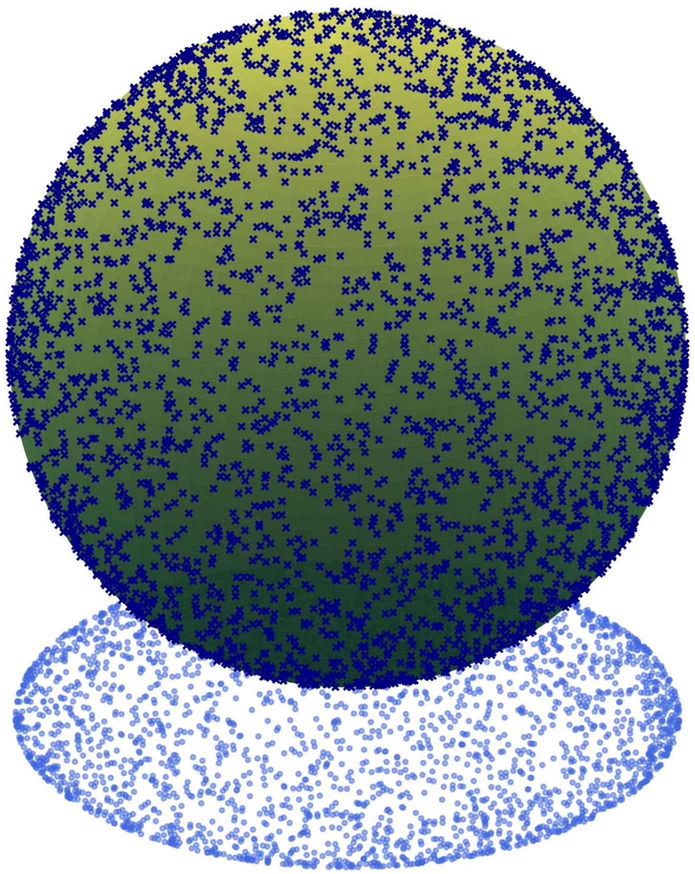
A hallmark of randomly drawn rotations is that they evenly distribute points on the sphere.

This is achieved by the Haar measure:

$$d\mu(\mathbf{n},\omega) = \frac{1}{4\pi^2}\left(1-\cos\omega\right)d\omega\, d\mathbf{n} \qquad \text{(axis n, angle } \omega\text{)}$$

So the distribution $f_1(\omega)$ of rotation angle is biased

$$\boxed{f_1(\omega)d\omega = \int_{\mathbf{n}\in S^2} d\mu = \frac{1}{\pi}\left(1-\cos\omega\right)d\omega}$$



Miles , Biometrika (1965)
Rummler, Math. Intelligencer (2002)

But powers of random rotation distribute uniformly

Repeating twice a rotation by ω_1 give a rotation by $2\omega_1$

folded onto $[0,\pi]$ by the symmetry $R(n,\omega)=R(-n,2\pi-\omega)$

$$\omega=\min\{2\omega_1,2\pi-2\omega_1\}$$

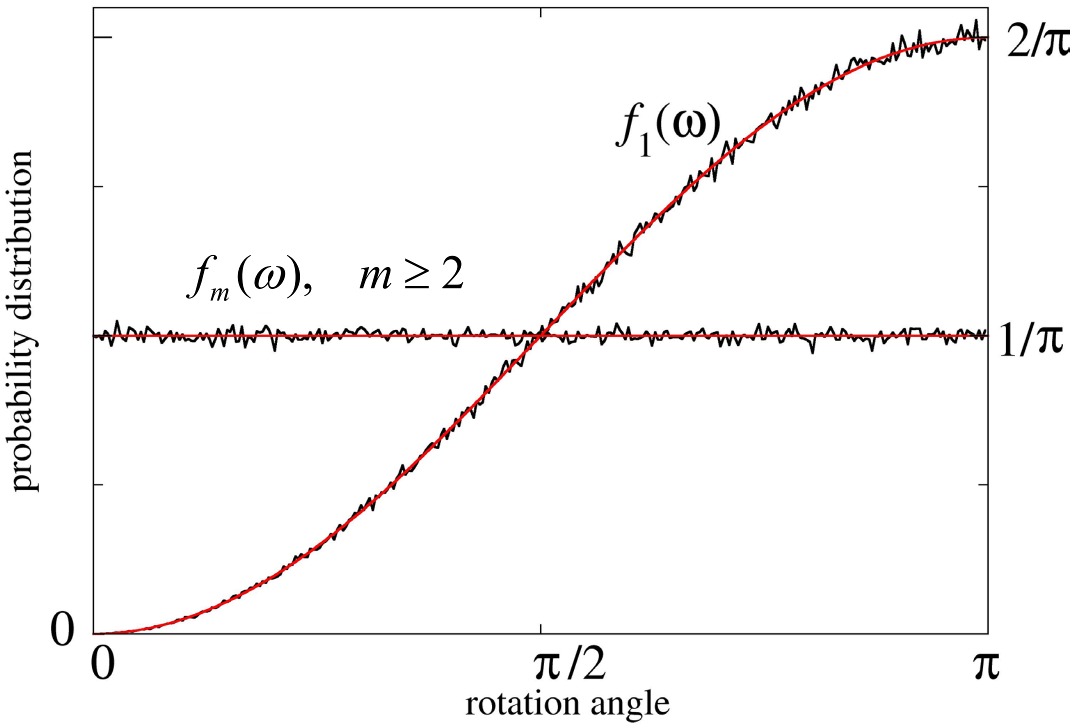
So the distribution $f_2(\omega)$ of doubled rotation angles is

$$f_2(\omega)=\tfrac{1}{2\pi}\left(1-\cos\tfrac{\omega}{2}\right)+\tfrac{1}{2\pi}\left(1-\cos\tfrac{2\pi-\omega}{2}\right)=\tfrac{1}{\pi}$$

A version of the Diaconis shuffling transition

for random walks in SO(3). For any $m\geq 2$,

$$\boxed{f_m(\omega)=\tfrac{1}{\pi}}$$



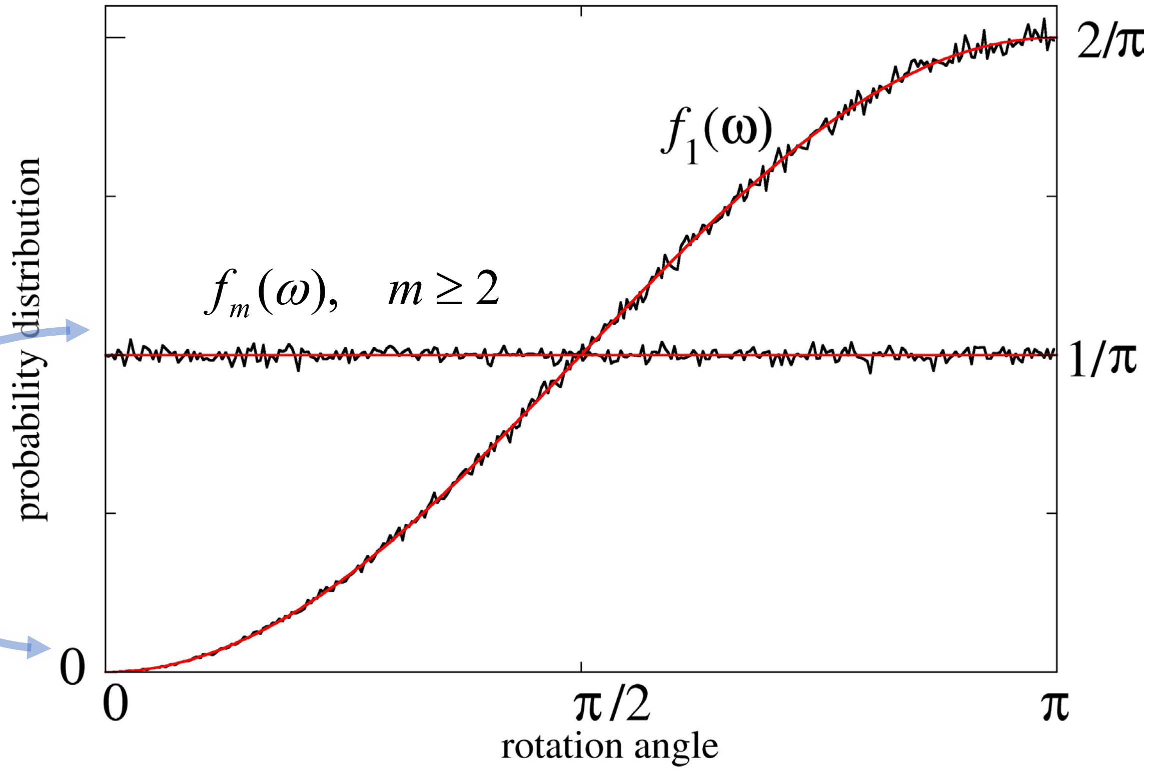
Diaconis & Shahshahan (1987)
Rains (2003)

Close to 1, there are very few random rotations,

but many powers R^m $(m \geq 2)$ of random rotations

Because there's only one identity $R(n,0)=1$

but a whole 2D manifold of identity roots $\left[R(n,\pi)\right]^2=1$



By varying the scaling factor λ, one samples the ensemble of random walks

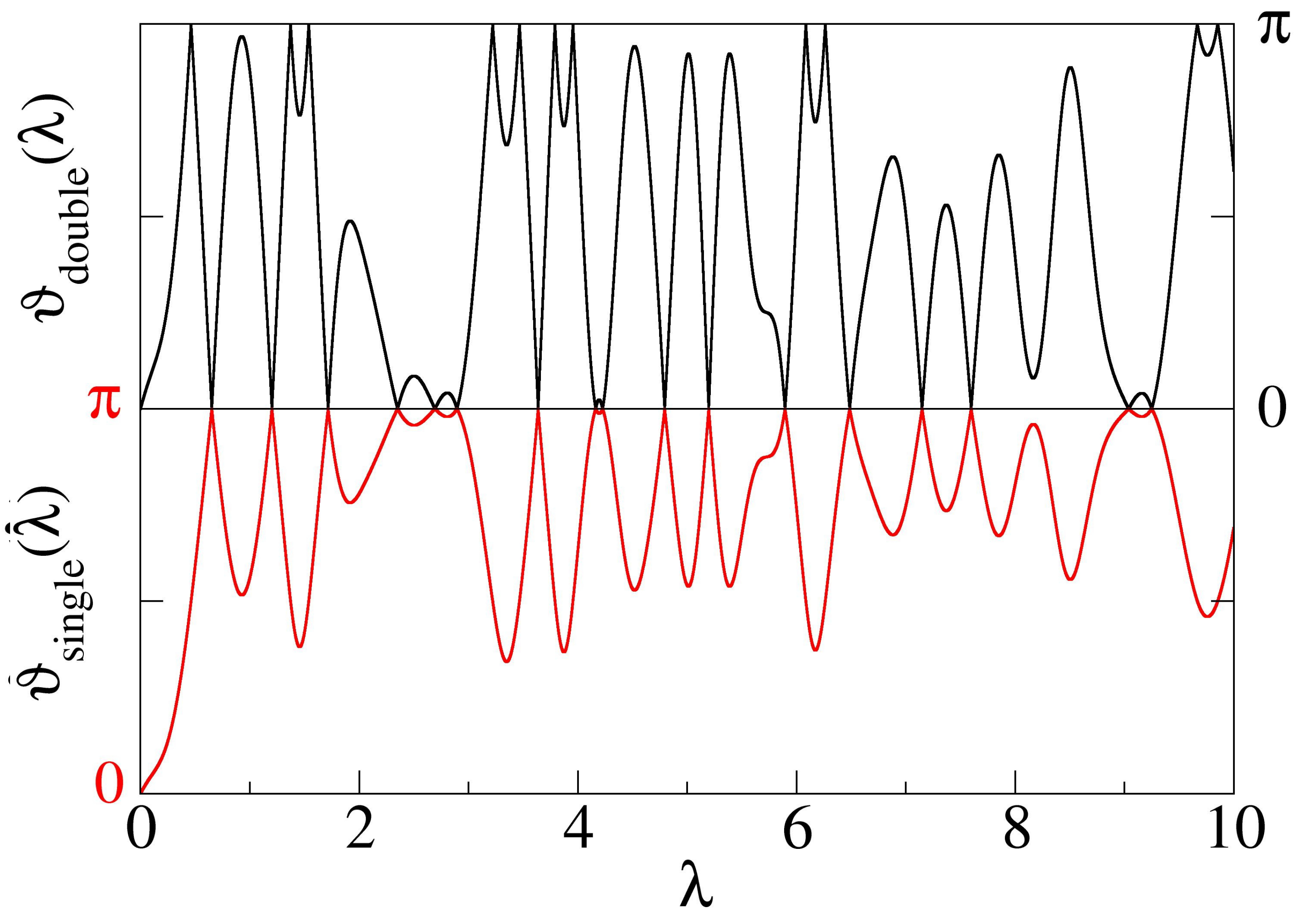
But double (or triple...) walks

have many solutions:

$$f_m(\omega) = \frac{1}{\pi}$$

Single walks almost never return home:

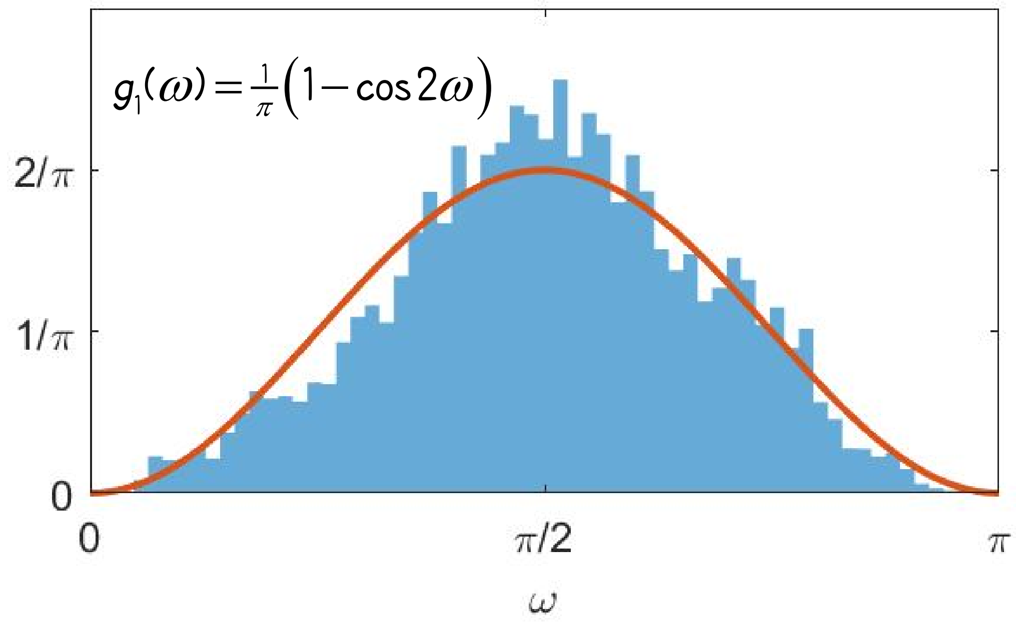
$$f_1(\omega) = \frac{1}{\pi}\left(1 - \cos\omega\right) \rightarrow \frac{1}{2\pi}\omega^2$$

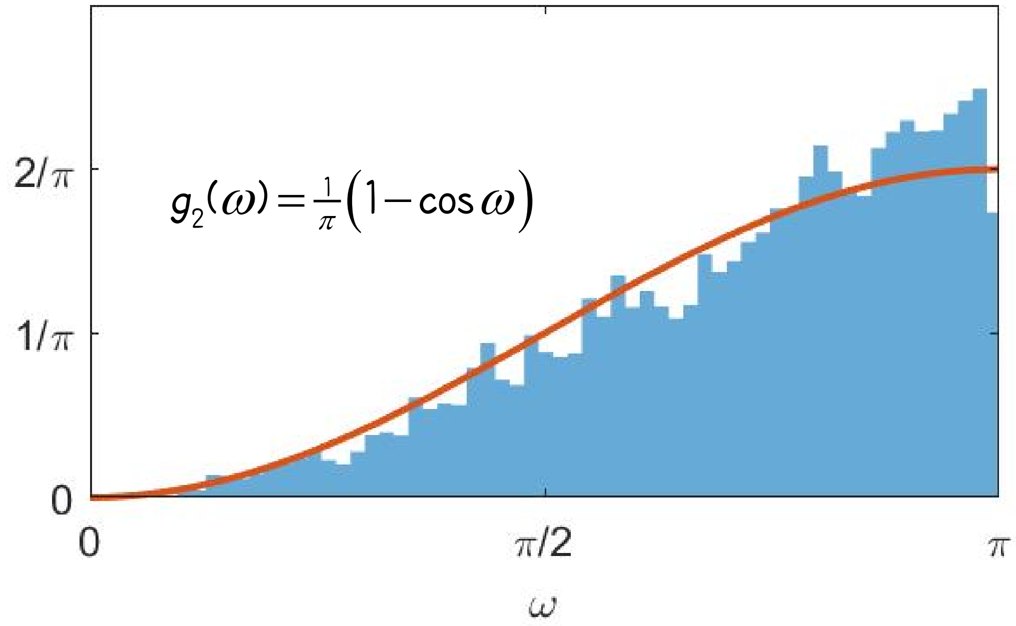


SU(2) shows a similar sharp threshold, only at m ≥ 3

$$g_m(\omega)=\begin{cases}\frac{1}{\pi}\left(1-\cos 2\omega\right) & : m=1\\ \frac{1}{\pi}\left(1-\cos\omega\right) & : m=2\\ \frac{1}{\pi} & : m=3\end{cases}$$

And varying λ (histogram) samples quite well the random ensemble





4. This was a probabilistic argument.

To show that scaling + doubling returns the system *precisely* to the origin, we use the multiplication rule of SU(2) (Rodrigues) and a Diophantine equation (Conway & Jones).

Eckmann et al. arXiv:2502.14367

THEOREM [Double walks returning to 1 are abundant]:

Let $\{\omega_j > 0, j=1,\ldots,N\}$ be the angles of rotation of N rotations R_j

Then, there is a $\lambda > 0$ for which $[W(\lambda)]^2 = [R_N^\lambda \cdot R_{N-1}^\lambda \cdot \ldots \cdot R_1^\lambda]^2 = 1.$

LEMMA [Single walks returning to 1 are rare]:

The identity can only be reached by a single walk

$W(\lambda) = R_N^\lambda \cdot R_{N-1}^\lambda \cdot \ldots \cdot R_1^\lambda = 1$ for some $\lambda > 0$

if all rotation axes are collinear, or if all rotation angles are commensurate, which are cases of negligible measure.

Except for trivial cases

(i) no rotation: $W(\lambda = 0) = 1.$

(ii) identity to begin with: $W(\lambda = 1) = 1.$

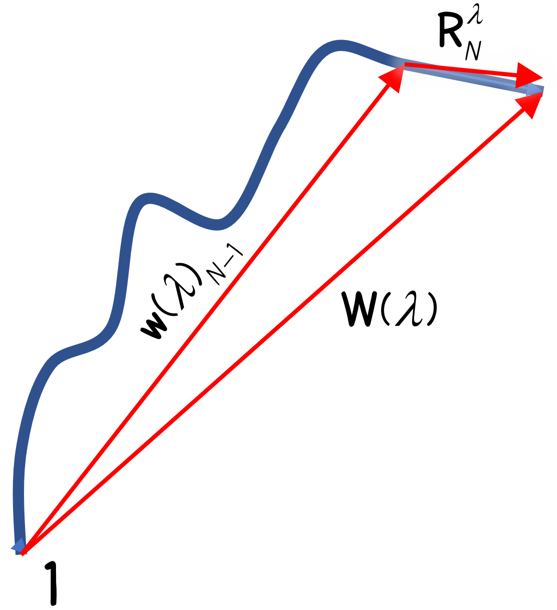
(iii) single rotation: $W(1) = R(n,\omega)$ so $W\left(\frac{2\pi}{\omega}\right) = 1.$

Rodrigues' formula for combining rotations is the multiplication rule of SU(2)

Definitions

- R_j, $j=1,\ldots,N$, are N rotations with rotation angles ω_j and axes $\mathbf{n}_j$,
- $\mathrm{W}(\lambda)=\mathrm{R}_N^\lambda\cdot\mathrm{R}_{N\text{-}1}^\lambda\cdot\ldots\cdot\mathrm{R}_1^\lambda$ is the scaled walk,
- $\mathrm{w}(\lambda)_j=\mathrm{R}_j^\lambda\cdot\mathrm{R}_{j\text{-}1}^\lambda\cdot\ldots\cdot\mathrm{R}_1^\lambda$ are partial scaled walks

  with total rotation angles $\phi(\lambda)_j$ and axes $\mathrm{a}(\lambda)_j$



PROOF of lemma (single walks returning to 1 are rare)

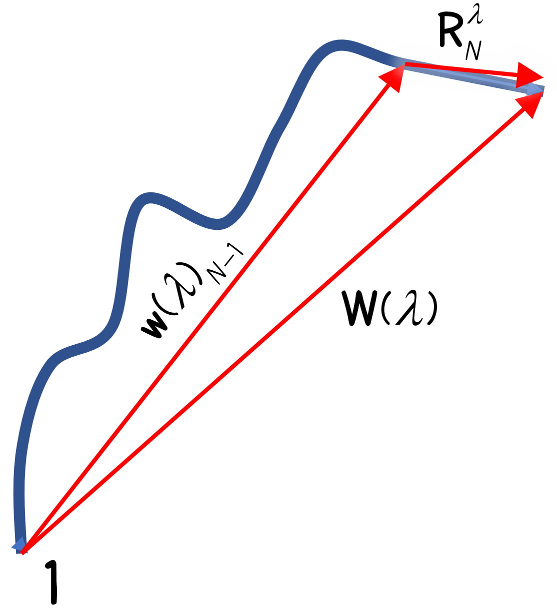
We recursively decompose the walk W. The first step $\mathrm{W}(\lambda)=\mathrm{R}_N^\lambda\cdot\mathrm{w}(\lambda)_{N-1}$:

$$\begin{aligned}\cos\tfrac{1}{2}\phi(\lambda)_N &= \cos\tfrac{1}{2}\phi(\lambda)_{N-1}\cos\tfrac{1}{2}\lambda\omega_N-\sin\tfrac{1}{2}\phi(\lambda)_{N-1}\sin\tfrac{1}{2}\lambda\omega_N\left[\mathbf{n}_N\cdot\mathbf{a}(\lambda)_{N-1}\right]\\ &= x(\lambda)_N\cos\left[\tfrac{1}{2}\left(\phi(\lambda)_{N-1}+\lambda\omega_N\right)\right]-\left[x(\lambda)_N-1\right]\cos\left[\tfrac{1}{2}\left(\phi(\lambda)_{N-1}-\lambda\omega_N\right)\right]\end{aligned}$$

where $x(\lambda)_k\equiv\tfrac{1}{2}\left[1+\mathbf{n}_k\cdot\mathbf{a}(\lambda)_{k-1}\right]$ so $0\le x(\lambda)_k\le 1$.

Olinde Rodrigues (1840)

$$\cos\tfrac{1}{2}\phi(\lambda)_N = \cos\tfrac{1}{2}\phi(\lambda)_{N-1}\cos\tfrac{1}{2}\lambda\omega_N - \sin\tfrac{1}{2}\phi(\lambda)_{N-1}\sin\tfrac{1}{2}\lambda\omega_N\left[\mathbf{n}_N\cdot\mathbf{a}(\lambda)_{N-1}\right]$$
$$= x(\lambda)_N\cos\left[\tfrac{1}{2}\left(\phi(\lambda)_{N-1}+\lambda\omega_N\right)\right]-\left[x(\lambda)_N-1\right]\cos\left[\tfrac{1}{2}\left(\phi(\lambda)_{N-1}-\lambda\omega_N\right)\right]$$



Now, if $W(\lambda)=1$, then $\phi(\lambda)_N=0$, and $\cos\tfrac{1}{2}\phi(\lambda)_N=1$. Two possibilities:

(i) The rotations are colinear, $\mathbf{n}_N\cdot\mathbf{a}(\lambda)_{N-1}=\pm1$, and $x(\lambda)_N\in\{0,1\}$.

This case is of measure 0 in choice of $\mathbf{a}(\lambda)_{N-1}$ once $\mathbf{n}_N$ is given.

(ii) Both cosines are 1, $\cos\left[\tfrac{1}{2}\left(\phi(\lambda)_{N-1}\pm\lambda\omega_N\right)\right]=1$.

So, $\tfrac{1}{2}\left(\phi(\lambda)_{N-1}+\lambda\omega_N\right)=2\pi\alpha_N$ and $\tfrac{1}{2}\left(\phi(\lambda)_{N-1}-\lambda\omega_N\right)=2\pi\beta_N$, with α_N,β_N integers.

Therefore, $\lambda\omega_N=2\pi\left(\alpha_N-\beta_N\right)$ and $\phi(\lambda)_{N-1}=2\pi\left(\alpha_k+\beta_k\right)$ $\Rightarrow\cos\tfrac{1}{2}\phi(\lambda)_{N-1}=1$.

Then, we continue the recursion and get for all $k=1,2,\ldots,N$

$$\lambda\omega_k=2\pi\left(\alpha_k-\beta_k\right),$$

that is all rotation angles ω_k are commensurate, which is of measure 0. □

The double walk theorem is a trigonometric Diophantine problem

THEOREM [Double walks returning to 1 are abundant]:

Let $\{\omega_j > 0, j=1,\ldots,N\}$ be the angles of rotation of N rotations R_j

Then, there is a $\lambda > 0$ for which $[W(\lambda)]^2 = [R_N^\lambda \cdot R_{N-1}^\lambda \cdot \ldots \cdot R_1^\lambda]^2 = 1.$

PROOF

Using Rodrigues formula recursively, we find

$$\cos\tfrac{1}{2}\phi(\lambda)_N = \prod_{j=i}^{N} \cos\tfrac{1}{2}\lambda\omega_j + \mathrm{Rem}.$$

where Rem is a sum of ℓ products that contain at least one sine, $\sin\tfrac{1}{2}\lambda\omega_j$

and factors <1 (dot and cross product of unit vectors).

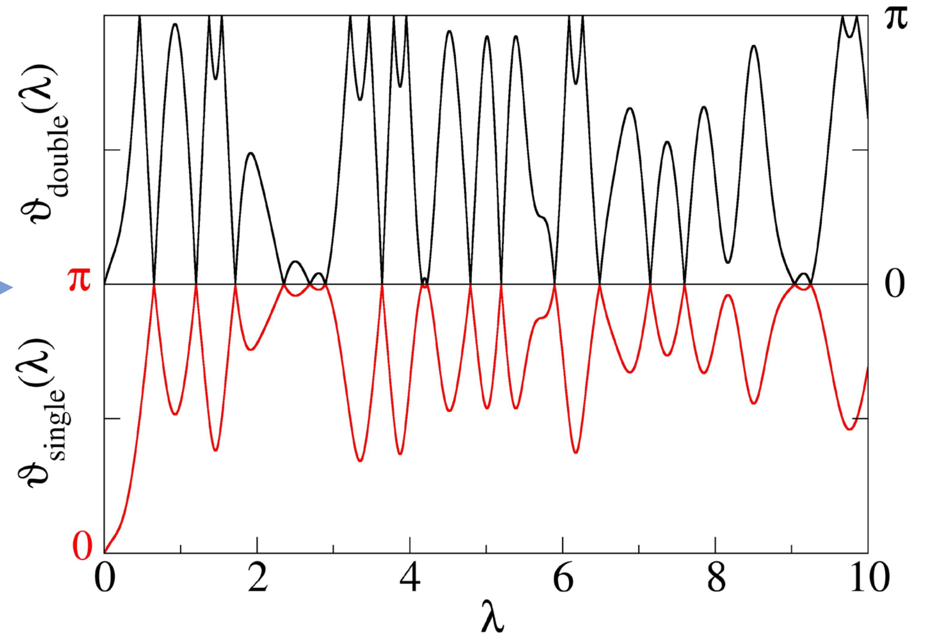
$$\cos\tfrac{1}{2}\phi(\lambda)_N = \prod_{j=i}^{N} \cos\tfrac{1}{2}\lambda\omega_j + \mathrm{Rem}$$

For the talk assume N is odd. Now, we have a Diophantine problem:

We look for $\lambda\_$ for which all $\cos\tfrac{1}{2}\lambda\_\omega_j \sim -1$ and all $\sin\tfrac{1}{2}\lambda\_\omega_j \sim 0$.

For this we need $\tfrac{1}{2}\lambda\omega_j \simeq n_j\pi$, with odd n_j.

$$\boxed{\tfrac{\lambda}{2\pi}\omega_j \equiv n_0\omega_j \simeq n_j \in \text{odd } \mathbb{Z}}$$



We can take $\left|n_0\omega_j - n_j\right| \le \dfrac{2\varepsilon}{\ell}$ and have $\cos\tfrac{1}{2}\phi(\lambda\_)_N = -1 + o(\varepsilon) < 0$ ($\ell =$ no. of terms in Rem)

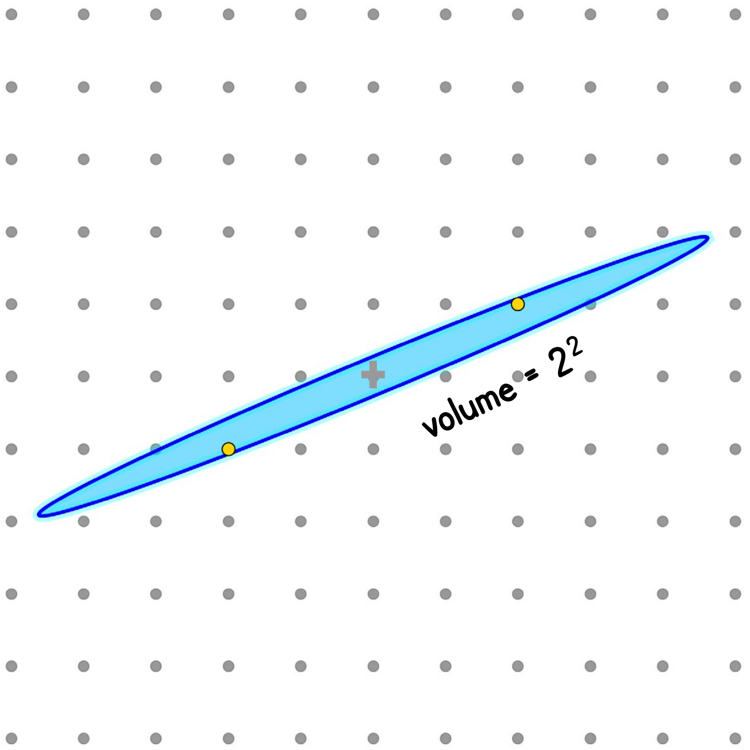
Since $\cos\tfrac{1}{2}\phi(0)_N = 1$, by continuity there is λ_* with $\cos\tfrac{1}{2}\phi(\lambda_*)_N = 0$.

There, $\tfrac{1}{2}\phi(\lambda_*)_N = \tfrac{\pi}{2}$ and $\phi(\lambda_*)_N = \pi$.

This solves the probelm since $\left[W(\lambda_*)\right]^2 = 1$.

The trigonometric Diophantine problem is solved using Minkowski's theorem

Minkowski's Theorem: Take a lattice in $\mathbb{R}^M$
and a set S convex and symmetric w.r.t. 0.
Then if $\text{Volume}(S) > 2^M \times \text{Volume}(\text{Cell})$
then S must contain a non-zero point of the lattice.



To find a solution in the lattice defined by $\frac{\lambda}{2\pi}\omega_j \equiv n_0\omega_j \simeq n_j \in \text{odd } \mathbb{Z}$,
we construct a convex "needle" symmetric w.r.t.
with a volume $2^{N+1} \times \text{Volume}(cell) = 2^{N+1} \times 2^N$

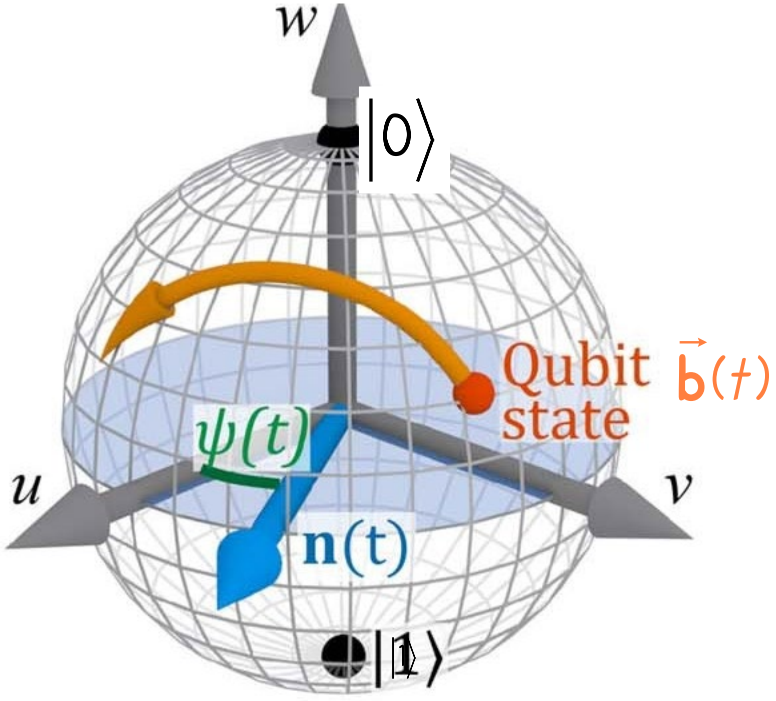
$$S = \left\{ \left(n_0, n_1, \ldots, n_N\right) \in \mathbb{R}^{1+N} \; : \; \left|n_0\right| \le \frac{\ell^N}{\varepsilon^N}, \; \left|n_0\omega_j - n_j\right| \le \frac{2\varepsilon}{\ell}, \; j = 1, \ldots, N \right\}.$$

($\ell =$ no. of terms in Rem)

This ensures the condition for $\cos\frac{1}{2}\phi(\lambda_-)_N < 0$ and therefore $\cos\frac{1}{2}\phi(\lambda_*)_N = 0$, so $\left[W(\lambda_*)\right]^2 = 1$. □

5. What is it good for?

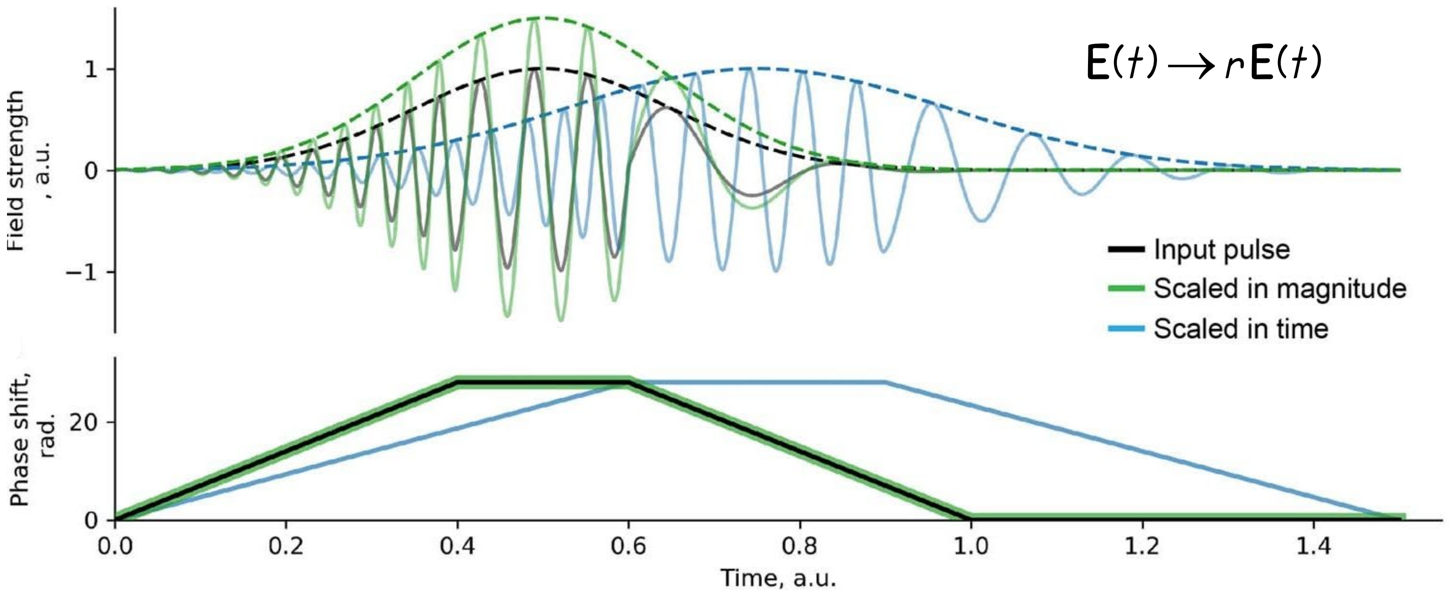
Rotation of a qubit in electric field can be rescaled to return to the original state by applying thetheorem



$$\left|\mathbf{b}(t)\right\rangle = \cos\frac{\theta}{2}\left|0\right\rangle + e^{i\phi}\sin\frac{\theta}{2}\left|1\right\rangle$$
$$\Leftrightarrow \vec{\mathbf{b}}(t) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

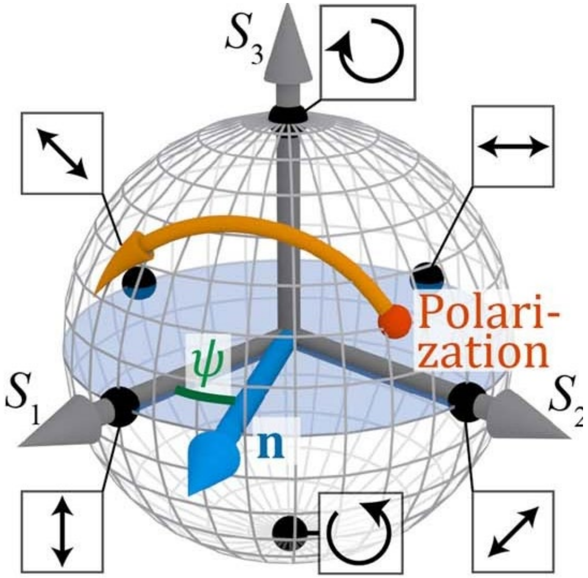
Bloch vector $\vec{\mathbf{b}}$ in electric field $E(t)\cos\left(\omega_0 t - \psi(t)\right)$ (resonance $\omega_0 = \frac{W}{\hbar}$)

turns at angular velocity $\mathbf{n}(t) = -\frac{\mu_{ab}}{\hbar}E(t)\left(\cos\psi(t), \sin\psi(t), 0\right)$.



$E(t) \rightarrow E(rt), \quad \psi(t) \rightarrow \psi(rt)$

Analogous application applies to

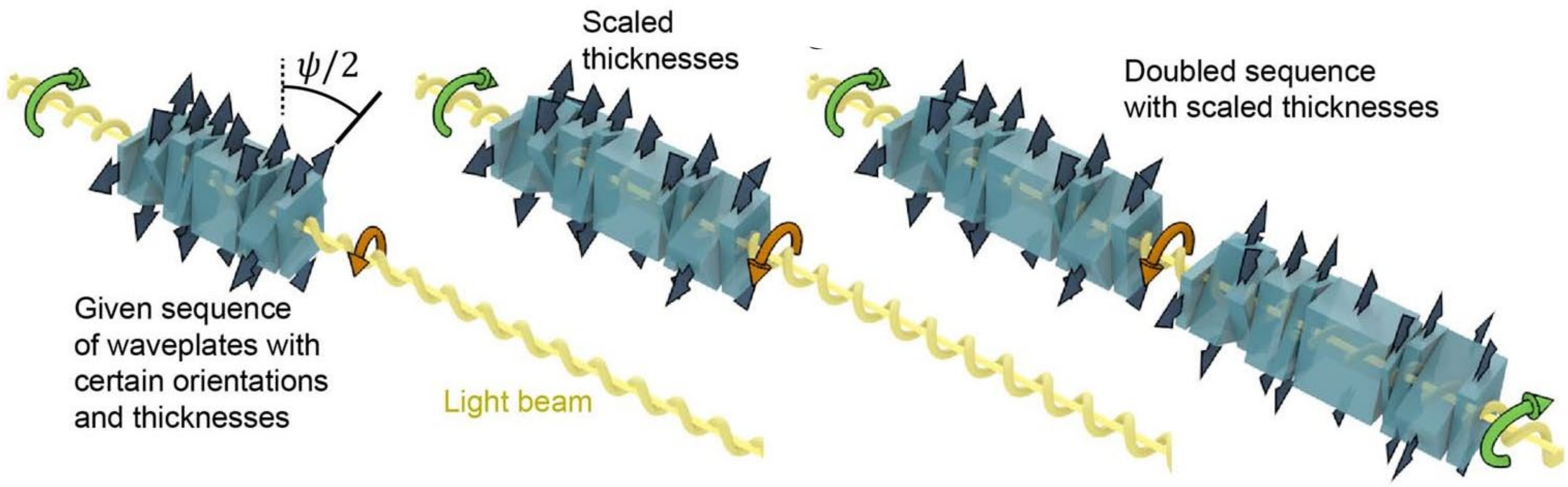


Light passing through 'almost any' stack of waveplates returns to original polarization if the stack is rescaled and doubled

Polarization vector of light passing a thin uniaxial waveplate rotates by an angle δt_i plate thickness around axis $\mathbf{n}(t) = \left(\cos\psi(t), \sin\psi(t), 0\right)$.

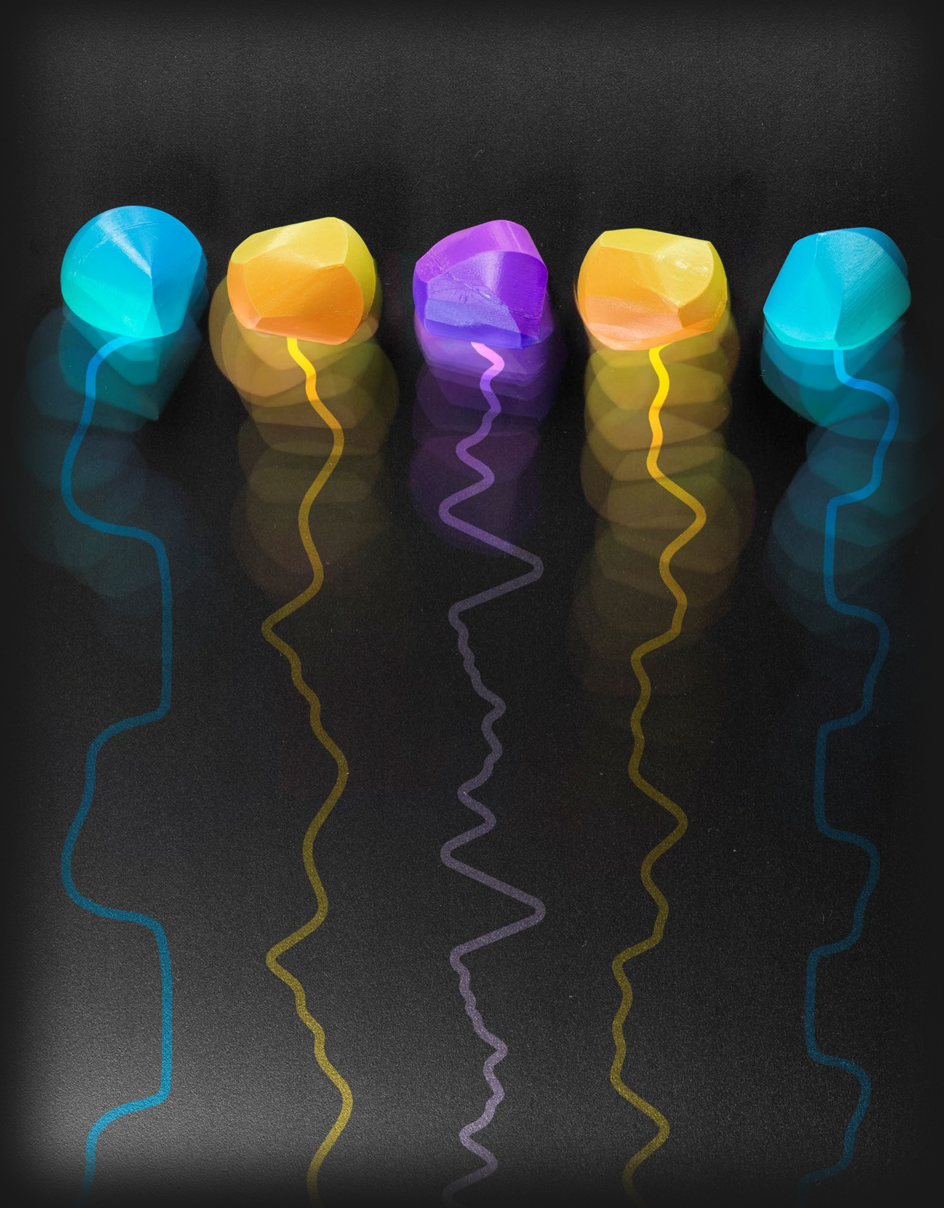
($\frac{1}{2}\psi =$ waveplate optical axis)

$$S_0 = I = E_x^2 + E_y^2$$
$$S_1 = Q = E_x^2 - E_y^2$$
$$S_2 = U = 2E_x E_y \cos\delta$$
$$S_3 = V = 2E_x E_y \sin\delta$$



THANKS

Yaroslav SOBOLEV • Jean-Pierre ECKMANN



Sobolev et al. Nature 2023; **Eckmann** et al. AMS Notices 2024;

Eckmann et al. Walks in Rotation Spaces Return Home When Doubled and Scaled. arXiv:2502.14367

FINITE PATH